

ON A TWISTED EULER-POINCARÉ PAIRING FOR GRADED AFFINE HECKE ALGEBRAS

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ABSTRACT. We study a twisted Euler-Poincaré pairing for graded affine Hecke algebras, and give a precise connection to the twisted elliptic pairing of Weyl groups defined by Ciubotaru-He [12]. The Ext-groups for an interesting class of parabolically induced modules are also studied in a connection with the twisted Euler-Poincaré pairing. We also study a certain space of graded Hecke algebra modules which equips with the twisted Euler-Poincaré pairing as an inner product.

1. INTRODUCTION

This paper studies a twisted Euler-Poincaré pairing on the space of virtual representations for the graded affine Hecke algebra. This twisted pairing is motivated from the twisted elliptic pairing of Weyl group recently developed by Ciubotaru-He [12], and we give a precise relations between these two pairings. In the same spirit as the Euler-Poincaré pairing of p -adic groups by Schneider-Stuhler [31] and others, an appropriate subspace of the virtual representations for the graded Hecke algebra is equipped with the twisted Euler-Poincaré pairing as an inner product. We shall discuss those twisted elliptic spaces defined by the twisted Euler-Poincaré pairing (based on several previous work by others [10], [11], [12], [25] and [30]).

In more detail, let (R, V, R^\vee, V^\vee) be a root data of a crystallographic type (Section 2.1) and let W be the finite reflection group acting on R . Let Δ be the set of simple roots. Let δ be an involution on the root system with $\delta(\Delta) = \Delta$. Then δ induces an involution on W which is still denoted by δ . A recent paper of Ciubotaru-He [12] defined the δ -twisted elliptic pairings on the representations U and U' of $W \rtimes \langle \delta \rangle$ as:

$$\langle U, U' \rangle_W^{\delta\text{-ellip}, V} = \frac{1}{|W|} \sum_{w \in W} \text{tr}_U(w\delta) \overline{\text{tr}_{U'}(w\delta)} \det_V(1 - w\delta),$$

where tr is the trace of w acting on U or U' . This twisted elliptic pairing is closely related to the Lusztig-Shoji algorithm.

When $\delta = \text{Id}$, the pairing coincides with the one defined by Reeder [25]. Suggested by Arthur [2] and verified by Reeder [25], a precise relation between the Euler-Poincaré pairing for p -adic groups and an elliptic pairing of Weyl groups was established. The goal of this paper is to study an analogue of the Euler-Poincaré pairing relating to the δ -twisted elliptic pairing considered by Ciubotaru-He. Our work is done in the level of graded affine Hecke algebra, which was introduced by Lusztig in [24] for the study of representations of p -adic groups and Iwahori-Hecke algebras.

Let \mathbb{H} be the graded affine Hecke algebra associated to a crystallographic root system (R, V, R^\vee, V^\vee) and a parameter function k (Definition 2.1). The action of δ can be extended to the Weyl group, and then extended to \mathbb{H} . For $\mathbb{H} \rtimes \langle \delta \rangle$ -modules X and Y , we define the δ -twisted Euler-Poincaré pairing on X and Y (regarded as \mathbb{H} -modules):

$$\mathrm{EP}_{\mathbb{H}}^{\delta}(X, Y) = \sum_i (-1)^i \mathrm{trace}(\delta^* : \mathrm{Ext}_{\mathbb{H}}^i(X, Y) \rightarrow \mathrm{Ext}_{\mathbb{H}}^i(X, Y)),$$

where Ext -groups are taken in the category of \mathbb{H} -modules. Here δ^* is a natural map induced from the action of δ on X and Y . Our first main result is the following:

Theorem 1.1. *(Proposition 3.4, Theorem 4.11) Suppose δ induces an inner automorphism on W (equivalently $\delta = \mathrm{Id}$ or δ arises from the longest element in the Weyl group (see 2.2)). For any finite dimensional $\mathbb{H} \rtimes \langle \delta \rangle$ -modules X and X' ,*

$$\mathrm{EP}_{\mathbb{H}}^{\delta}(X, X') = \langle \mathrm{Res}_W X, \mathrm{Res}_W X' \rangle_W^{\delta\text{-ellip}, V},$$

where Res_W is the restriction to the W -representation.

Theorem 1.1 for $\delta = \mathrm{Id}$ was established by Reeder [25] for equal parameter cases, and was independently proved by Opdam-Solleveld [28] for arbitrary parameters (in different settings). Nevertheless, our approach in proving Theorem 1.1 is independent from their work, and is self-contained. We remark our proof of Theorem 1.1 also holds for non-crystallographic cases, and the consequences for those cases will be considered elsewhere.

Our study begins with the construction of an explicit projective resolution on \mathbb{H} -modules. The idea of the construction came from the standard Koszul resolution. A remarkable point is that taking the Hom -functor on the resolution, the Hom -spaces between \mathbb{H} -modules are turned into Hom -spaces between Weyl group representations via Frobenius reciprocity, which is also essential in the proof of Theorem 1.1.

When $\delta = \mathrm{Id}$, the pairing defines an inner product on a subspace of the \mathbb{H} -representation ring. This space has been known and studied in [25] and [28]. Our focus of the remaining discussion will be on the case that δ is the automorphism θ arising from the longest element in the Weyl group (see (2.2)). Similar to the case for $\theta = \mathrm{Id}$, an appropriate subspace of the representation ring of \mathbb{H} is equipped with $\mathrm{EP}_{\mathbb{H}}^{\theta}$ as an inner product. We call such space to be θ -twisted elliptic as an analogue to the case in p -adic groups considered by Schneider-Stuhler [31]. Such θ -twisted elliptic space can also be regarded as the elliptic representation space of $\mathbb{H} \rtimes \langle \theta \rangle$. We shall describe those θ -twisted elliptic space in the next paragraph.

Let $\mathcal{N}_{\mathrm{sol}}$ be the set of nilpotent elements which have a solvable centralizer in the related Lie algebra to the root system. This set naturally arises from the study of the spin representations of Weyl groups as well as the Dirac cohomology for the graded affine Hecke algebra ([12], [4], [8], [10]). In particular, the work of Ciubotaru-He [12] implies that in the case of equal parameters, the θ -twisted elliptic representation space of \mathbb{H} is spanned by tempered modules which correspond to a nilpotent element in $\mathcal{N}_{\mathrm{sol}}$ under the Kazhdan-Lusztig parametrization (Theorem 6.4). For the simplicity later, we shall call those tempered modules to be solvable.

Those solvable tempered modules can be divided into three classes. The first ones are those (ordinary) elliptic tempered modules (in the sense of Reeder [25]). The second ones are those irreducible non-elliptic tempered modules which are not properly parabolically induced. This happens for the type D_n for n odd and $n \geq 9$ (see Remark 6.7). The third ones are certain irreducible, tempered and parabolically induced modules. It turns out that those irreducible tempered module in the third class can be characterized by a simple condition on the parabolic subalgebra which it is induced from. Those classes of modules are called rigid modules in Definition 5.1 and Proposition 6.6. A deeper reasoning for such condition indeed comes from the Plancherel measure and R -groups (in the sense of Opdam [26] and [16] respectively). The study related to those harmonic analysis interpretations on solvable tempered modules will be carried out elsewhere [9] (also see Remark 6.8).

Our second part of the paper is to study the Ext-groups on the rigid modules in Definition 5.1. (See Remark 5.2 for more comments on the terminology.) As mentioned above, rigid modules provide most examples of solvable tempered modules which are not elliptic. In other words, they lie in the radical of the (ordinary) Euler-Poincaré pairing, but not in the radical of the twisted Euler-Poincaré pairing. Then it is natural to ask how those rigid modules behave differently under the two pairings via a study of the Ext-groups and the θ^* -action.

Another main result in this paper is Theorem 1.2 below.

Theorem 1.2. *(Theorem 5.15) Let \mathbb{H} be the graded affine Hecke algebra associated to a crystallographic root system and a parameter function k (Definition 2.1). Let X be a rigid of discrete series of \mathbb{H} (Definition 5.1). Then*

$$\dim \operatorname{Ext}_{\mathbb{H}}^i(X, X) = \binom{r}{i} = \frac{r!}{(r-i)!i!}, \quad \text{for } i \leq r$$

for some fixed r (which is described precisely in Theorem 5.15). Furthermore θ^ acts on $\operatorname{Ext}_{\mathbb{H}}^i(X, X)$ by the multiplication of a scalar of $(-1)^i$.*

We remark that our computation of Ext-groups in Theorem 1.2 essentially uses the Ext-groups for discrete series from the work of Delorme-Opdam [15] and Opdam-Solleveld [28]. Apart from the deep analytic result from [15] and [28], the main tool of our computation is the projective resolution developed in Section 3 with some careful analysis on the structure of rigid modules. It is possible to apply our techniques to other tempered modules, but results obtained by current approach is more complete for those rigid modules.

The approach used in this paper to study Ext-groups differs from the one used by Adler-Prasad [1] for p -adic groups and the one by Opdam-Solleveld [30] for affine Hecke algebras, and so we hope our study provides another perspective on the extensions of representations. Our approach should also be applicable for the study of the graded Hecke algebra of a noncrystallographic type and other similar algebraic structure such as the degenerate affine Hecke-Clifford algebra.

We briefly outline the organization of this paper. Section 2 is to define and review several important objects such as the map θ , graded affine Hecke algebras and tempered modules.

In Section 3, we construct an explicit projective resolution of an \mathbb{H} -module, which is the main tool in this paper. In Section 4, we define the twisted Euler-Poincaré pairing and prove Theorem 1.1. Section 5 is devoted to compute the θ^* action on some Ext-groups of certain modules. Section 6 is to study and describe the twisted elliptic space in terms of the Kazhdan-Lusztig model.

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2. PRELIMINARIES

2.1. Root systems and basic notations. Let R be a reduced root system of a crystallographic type. Let Δ be a fixed choice of simple roots in R . Then Δ determines the set of positive roots R^+ . Let W be the finite reflection group of R . Let V'_0 be the real space spanned by Δ and let V_0 be a real vector space containing V'_0 as a subspace. For any $\alpha \in \Delta$, let s_α be the simple reflection in W associated to α (i.e. $\alpha \in V_0$ is in the -1 -eigenspace of s_α). For $\alpha \in R$, let $\alpha^\vee \in \text{Hom}_{\mathbb{R}}(V_0, \mathbb{R})$ such that

$$s_\alpha(v) = v - \langle v, \alpha^\vee \rangle \alpha,$$

where $\langle v, \alpha^\vee \rangle = \alpha^\vee(v)$. Let $R^\vee \subset \text{Hom}_{\mathbb{R}}(V_0, \mathbb{R})$ be the collection of all α^\vee . Let $V_0^\vee = \text{Hom}_{\mathbb{R}}(V_0, \mathbb{R})$.

By extending the scalars, let $V = \mathbb{C} \otimes_{\mathbb{R}} V_0$ and let $V^\vee = \mathbb{C} \otimes_{\mathbb{R}} V_0^\vee$. We call (R, V, R^\vee, V^\vee) to be a root datum.

For any subset J of Δ , define V_J to be the complex subspace of V spanned by simple roots in J . Let $R_J = V_J \cap R$. Let $R_J^\vee = \{\alpha^\vee \in R^\vee : \alpha \in R_J\}$. Let V_J^\vee be the subspace of V^\vee spanned by the coroots in R_J^\vee . Let W_J be the subgroup of W generated by the elements s_α for $\alpha \in J$. Define

$$V_J^\perp = \{v \in V : \langle v, v_1^\vee \rangle = 0 \text{ for all } v_1^\vee \in V_J^\vee\},$$

and

$$(V_J^\vee)^\perp = \{v^\vee \in V^\vee : \langle v_1, v^\vee \rangle = 0 \text{ for all } v_1 \in V_J\}.$$

Let $J \subset \Delta$. Let $w_{0,J}$ be the longest element in W_J . When $J = \Delta$, we simply write w_0 for $w_{0,\Delta}$. Let W^J be the set of minimal representatives in the cosets in W/W_J . Let w_0^J be the longest element in W^J .

2.2. Graded affine Hecke algebras. Let $k : \Delta \rightarrow \mathbb{R}$ be a parameter function such that $k(\alpha) = k(\alpha')$ if α and α' are in the same W -orbit. We shall simply write k_α for $k(\alpha)$.

Definition 2.1. [24, Section 4] The graded affine Hecke algebra $\mathbb{H} = \mathbb{H}_W$ associated to a root data (R, V, R^\vee, V^\vee) and a parameter function k is an associative algebra with an unit

over \mathbb{C} generated by the symbols $\{t_w : w \in W\}$ and $\{f_w : w \in V\}$ satisfying the following relations:

- (1) The map $w \mapsto t_w$ from $\mathbb{C}[W] = \bigoplus_{w \in W} \mathbb{C}w \rightarrow \mathbb{H}$ is an algebra injection,
- (2) The map $v \mapsto f_v$ from $S(V) \rightarrow \mathbb{H}$ is an algebra injection,

For simplicity, we shall simply write v for f_v from now on.

- (3) the generators satisfy the following relation:

$$t_{s_\alpha} v - s_\alpha(v) t_{s_\alpha} = k_\alpha \langle v, \alpha^\vee \rangle.$$

Notation 2.2. Let $J \subset \Delta$. Define \mathbb{H}_J to be the subalgebra of \mathbb{H} generated by all $v \in V$ and t_w ($w \in W_J$). We also define $\overline{\mathbb{H}}_J$ to be the subalgebra of \mathbb{H} generated by all $v \in V_J$ and t_w ($w \in W_J$). Here V_J and W_J is defined in Section 2.1. Note that \mathbb{H}_J decomposes as

$$\mathbb{H}_J = \overline{\mathbb{H}}_J \otimes S(V_J^\perp).$$

Note that \mathbb{H}_J is the graded affine Hecke algebra associated to the root data $(R, V_0, R^\vee, V_0^\vee)$ and $\overline{\mathbb{H}}_J$ is the graded affine Hecke algebra associated to the root data $(R, V_J, R^\vee, V_J^\vee)$.

Notation 2.3. According to (1) and (2), we shall view $\mathbb{C}[W]$ and $S(V)$ as the natural subalgebras of \mathbb{H} . For an \mathbb{H} -module X (resp. \mathbb{H}_J -module X with $J \subset \Delta$), denote $\text{Res}_W X$ (resp. $\text{Res}_{W_J} X$) be the restriction of X to a $\mathbb{C}[W]$ -module (resp. $\mathbb{C}[W_J]$ -module). $\text{Res}_{\mathbb{H}_J}$ and $\text{Res}_{\overline{\mathbb{H}}_J}$ are defined similarly for \mathbb{H} -modules.

For $v \in V$, we define the following element in \mathbb{H} :

$$(2.1) \quad \widetilde{v} = v - \frac{1}{2} \sum_{\alpha \in R^+} c_\alpha \langle v, \alpha^\vee \rangle s_\alpha.$$

This element is used in [4] for the study of the Dirac cohomology for graded affine Hecke algebras.

Lemma 2.4. For any $w \in W$ and $v \in V$, $t_w \widetilde{v} = \widetilde{w(v)} t_w$.

Proof. It suffices to show for the case that w is a simple reflection $s_\beta \in W$.

$$\begin{aligned} t_{s_\beta} \widetilde{v} &= t_{s_\beta} \left(v - \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \langle v, \alpha^\vee \rangle t_{s_\alpha} \right) \\ &= s_\beta(v) t_{s_\beta} + k_\beta \langle v, \beta^\vee \rangle - \frac{1}{2} k_\beta \langle v, \beta^\vee \rangle - \frac{1}{2} \sum_{\alpha \in R^+ \setminus \{\beta\}} k_\alpha \langle v, \alpha^\vee \rangle t_{s_\beta(\alpha)} \\ &= s_\beta(v) t_{s_\beta} - \frac{1}{2} k_\beta \langle v, s_\beta(\beta^\vee) \rangle - \frac{1}{2} \sum_{\alpha \in R^+ \setminus \{\beta\}} k_\alpha \langle v, s_\beta(\alpha^\vee) \rangle t_\alpha t_{s_\beta} \\ &= s_\beta(v) t_{s_\beta} - \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \langle s_\beta(v), \alpha^\vee \rangle t_\alpha t_{s_\beta} \\ &= \widetilde{s_\beta(v)} \end{aligned}$$

□

2.3. Central characters of \mathbb{H} . The center of \mathbb{H} can be explicitly described as below.

Proposition 2.5. [24, Proposition 4.5] *The center of \mathbb{H} is equal to $S(V)^W$, where $S(V)^W$ is the set of the W -invariant polynomials in $S(V)$*

Definition 2.6. Let $Z(\mathbb{H})$ be the center of \mathbb{H} . The central character of an irreducible \mathbb{H} -module X is the map $\chi : Z(\mathbb{H}) \rightarrow \mathbb{C}$ such that $\chi(z)$ is the scalar that z acts on X .

According to Proposition 2.5, the central character χ can be parametrized by the W -orbits $[v]$ in V such that

$$\chi(z) = z(v),$$

where v is a representative of the W -orbit $[v]$ and $z(v)$ is regarded as the value of the polynomial z evaluated at v .

2.4. *-operation and *-Hermitian modules. We first define an anti-involutive *-operation which naturally comes from the p -adic groups as follow:

$$t_w^* = t_w^{-1} \quad \text{for } w \in W, \quad v^* = -t_{w_0} \overline{w_0(v)} t_{w_0}^{-1} = -v + \frac{1}{2} \sum_{\alpha \in R^+} \langle v, \alpha^\vee \rangle t_{s_\alpha}.$$

Here \bar{h} denotes the complex conjugation on h .

Definition 2.7. Let X be an \mathbb{H} -module. A function $f : X \rightarrow \mathbb{C}$ is said to be conjugate-linear if $f(\lambda x_1 + x_2) = \bar{\lambda} f(x_1) + f(x_2)$ for all $\lambda \in \mathbb{C}$ and $x_1, x_2 \in X$. The **-Hermitian dual* of X , denoted X^* , is the space of all the conjugate-linear functions $f : X \rightarrow \mathbb{C}$ equipped with the \mathbb{H} -action given by

$$(h.f)(x) = f(h^*.x) \quad \text{for all } x \in X.$$

It is straightforward to verify that the above \mathbb{H} -action is well-defined. An \mathbb{H} -module X is said to be **-Hermitian* if X is isomorphic to its Hermitian dual, or equivalently there exists a non-degenerate Hermitian form on X such that $\langle h.x_1, x_2 \rangle = \langle x_1, h^*.x_2 \rangle$ for all $h \in \mathbb{H}$ and $x_1, x_2 \in X$.

We say that X is **-unitary* if there exists a non-degenerate and positive-definite Hermitian form on X such that $\langle h.x_1, x_2 \rangle = \langle x_1, h^*.x_2 \rangle$ for all $h \in \mathbb{H}$ and $x_1, x_2 \in X$.

2.5. θ -action. Let θ be an involution on \mathbb{H} characterized by

$$(2.2) \quad \theta(v) = -w_0(v) \text{ for any } v \in V, \text{ and } \quad \theta(t_w) = t_{w_0 w w_0^{-1}} \text{ for any } w \in W,$$

where w_0 acts on v as the reflection representation of W .

Lemma 2.8. *For any $v \in V$, $\theta(\widetilde{v}) = \widetilde{\theta(v)}$.*

Proof. This follows from a straightforward computation.

□

Definition 2.9. For an \mathbb{H} -module X , define X^θ to be the \mathbb{H} -module such that X^θ is isomorphic to X as vector spaces and the \mathbb{H} -action is determined by:

$$\pi_{X^\theta}(h)x = \pi_X(\theta(h))x,$$

where π_X and π_{X^θ} are the maps defining the action of \mathbb{H} on X and X^θ respectively.

Definition 2.10. Let $J \subset \Delta$. For an \mathbb{H}_J -module X , we say $\gamma \in V^\vee$ is a *weight* of X if there exists a non-zero $x \in X$ such that $(v - \gamma(v))^k x = 0$ for all $v \in V$ and for some positive integer k . We call such x to be the *generalized weight vector* of γ .

Proposition 2.11. *Let X be an irreducible \mathbb{H} -module with a real central character. Assume that X satisfy one of the following conditions:*

- (1) *the central character of X is non-zero,*
- (2) *the parameter function k is identically equal to zero,*
- (3) *$k_\alpha \neq 0$ for all $\alpha \in \Delta$.*

Then X^θ is the Hermitian dual of X .

Proof. We sketch the proof. Let x_γ be a generalized weight vector of X of a weight $\gamma \in V^\vee$. Then for sufficiently large k and $v \in V_0$,

$$((v - \theta(\gamma)(v))^k \cdot f)(t_{w_0} \cdot x_\gamma) = f(t_{w_0}(\theta(v) - \theta(\gamma)(v))^k \cdot x_\gamma) = 0.$$

Hence $\overline{\theta(\gamma)} = \theta(\gamma)$ is a weight of the Hermitian dual of X . Then have the same weights.

If X satisfies (1), then the arguments in the proof of [3, Proposition 4.3.1] (also see [17, Theorem 5.5]) implies that X and X^θ are isomorphic. We now assume (1) does not hold for X . Then the central character of X is zero. If X satisfies (2), then the restriction of X to $\mathbb{C}[W]$ is an irreducible W -representation. Then it is easy to show that the Hermitian dual of X and X^θ are isomorphic. We now assume X satisfies (3). Then by [27, Theorem 1.3] or [23, Proposition 2.9], $\text{Ind}_{S(V)}^{\mathbb{H}} \mathbb{C}_0$ is irreducible and hence there is only one irreducible \mathbb{H} -module with the central character 0. This implies the Hermitian dual of X and X^θ are isomorphic.

□

Remark 2.12. We believe that Proposition 2.11 is true for all the \mathbb{H} -modules with a real central character (without assuming any one of the three conditions in the proposition). An evidence is that the Hermitian dual of X and X^θ have the same $S(V)$ and $\mathbb{C}[W]$ module structure. However, the author does not succeed to find a simple proof. For the purpose of this paper, modules satisfying any one of the three conditions suffice.

Corollary 2.13. *Let X be an irreducible \mathbb{H} -module with a real central character. Assume X satisfies any one of the three conditions in Proposition 2.11. Then X is a $*$ -Hermitian \mathbb{H} -module if and only if X and X^θ are isomorphic.*

2.6. Tempered modules and discrete series. Tempered modules and discrete series will be studied in Section 5 and 6. They provide the main examples of \mathbb{H} -modules X with the property $X^\theta = X$.

Definition 2.14. Recall that \mathbb{H} is associated to the root data (R, V, R^\vee, V^\vee) . An \mathbb{H} -module X is said to be *tempered* if for any weight $\gamma \in V^\vee$ of X , $\operatorname{Re}\langle \omega_\alpha, \gamma \rangle \leq 0$ for any fundamental weight ω_α in V . Here $\operatorname{Re}(a)$ denotes the real part of a complex number.

An \mathbb{H} -module is said to be a *discrete series* if X is tempered and all the inequalities in the definition of tempered modules are strict.

Theorem 2.15. [32, Theorem 7.2] *All irreducible discrete series has a real central character and are $*$ -unitary.*

Notation 2.16. Let Ξ be the set of triples (J, U, ν) such that $J \subset \Delta$, U is a $\overline{\mathbb{H}}_J$ -discrete series, and $\nu \in V_J^\vee$. For any $(J, U, \nu) \in \Xi$, denote $X(J, U, \nu)$ to be the parabolically induced module $\operatorname{Ind}_{\mathbb{H}_J}^{\mathbb{H}}(U \otimes \mathbb{C}_\nu) := \mathbb{H} \otimes_{\mathbb{H}_J} (U \otimes \mathbb{C}_\nu)$. When $\nu = 0$, we shall simply write $X(J, U)$ instead of $X(J, U, 0)$. We indeed consider $\nu = 0$ most of time in this paper. We call $X(J, U)$ to have a real central character (c.f. Theorem 2.15).

Proposition 2.17. [5, Corollary 1.4] *Let $(J, U, \nu) \in \Xi$. Then there exists a non-degenerate positive-definite $*$ -Hermitian form \langle, \rangle on $X(J, U, \nu)$ i.e. $\langle h.x, x' \rangle = \langle x, h^*.x' \rangle$. In particular, $X(J, U, \nu)$ is $*$ -unitary.*

Proof. This is [5, Corollary 1.4]. Since U is an irreducible \mathbb{H}_J -discrete series, Theorem 2.15 implies that there exists a non-degenerate $*$ -Hermitian form \langle, \rangle_J on U . Define a projection map $\operatorname{pr} : \mathbb{H} \rightarrow \mathbb{H}_J$ as follow: for $h \in \mathbb{H}$, h can be uniquely written as the form $\sum_{w \in W^J} t_w h_w$, where $h_w \in \mathbb{H}_J$. Then pr is defined as $\operatorname{pr}(h) = h_e$, where e corresponds to the trivial coset in W/W_J . Define the non-degenerate form \langle, \rangle on $X(J, U)$ as

$$\langle h_1 \otimes u_1, h_2 \otimes u_2 \rangle = \langle u_1, \operatorname{pr}(h_1^* h_2) u_2 \rangle_J.$$

It remains to verify \langle, \rangle satisfies the desired properties. □

We shall use the following result later:

Corollary 2.18. *Let $(J, U, 0) \in \Xi$. Suppose $X(J, U)$ satisfy one of the three conditions in Proposition 2.11. Then $X(J, U)$ is isomorphic to $X(J, U)^\theta$ as \mathbb{H} -modules.*

Proof. By Proposition 2.17, $X(J, U)$ is the direct sum of irreducible $*$ -Hermitian modules. Then the statement is a consequence of Proposition 2.17 and Corollary 2.13. □

2.7. $\text{Ext}_{\mathbb{H}}$ -groups. The following result about $\text{Ext}_{\mathbb{H}}$ -groups will be used several times later. Here $\text{Ext}_{\mathbb{H}}$ -groups are taken in the category of \mathbb{H} -modules.

Theorem 2.19. *Let X and Y be \mathbb{H} -modules. Then if X and Y have distinct central characters, then $\text{Ext}_{\mathbb{H}}^i(X, Y) = 0$ for all i .*

Proof. See for example [6, Theorem I. 4.1], whose proof can be modified to our setting. \square

3. A KOSZUL TYPE RESOLUTION ON \mathbb{H} -MODULES

We keep using the notation in Section 2.

3.1. Koszul-type resolution on \mathbb{H} -modules. Let X be an \mathbb{H} -module. Define a sequence of \mathbb{H} -module maps d_i as follows:

(3.3)

$$0 \rightarrow \mathbb{H} \otimes_{\mathbb{C}[W]} (\text{Res}_W X \otimes \wedge^n V) \xrightarrow{d_n} \dots \xrightarrow{d_{i+1}} \mathbb{H} \otimes_{\mathbb{C}[W]} (\text{Res}_W X \otimes \wedge^i V) \xrightarrow{d_i} \dots \xrightarrow{d_1} \mathbb{H} \otimes_{\mathbb{C}[W]} \text{Res}_W X \xrightarrow{d_0} X \rightarrow 0$$

such that $d_0 : \mathbb{H} \otimes X \rightarrow X$ given by

$$d_0(h \otimes x) = h.x$$

and for $i \geq 1$, $d_i : \mathbb{H} \otimes_{\mathbb{C}[W]} (\text{Res}_W X \otimes \wedge^i V) \rightarrow \mathbb{H} \otimes_{\mathbb{C}[W]} (\text{Res}_W X \otimes \wedge^{i-1} V)$ given by

$$(3.4) \quad d_i(h \otimes (x \otimes v_1 \wedge \dots \wedge v_i))$$

$$(3.5) \quad = \sum_{j=0}^i (-1)^j (h v_j \otimes x \otimes v_1 \wedge \dots \wedge \hat{v}_j \wedge \dots \wedge v_i - h \otimes v_j.x \otimes v_1 \wedge \dots \wedge \hat{v}_j \wedge \dots \wedge v_i).$$

Proposition 3.1. *The above d_i are well-defined maps and $d^2 = 0$ i.e. (3.3) is a well-defined complex.*

Proof. We proceed by an induction on i . It is easy to see that d_0 is well-defined. We now assume $i \geq 1$. To show d_i is independent of the choice of a representative in $\mathbb{H} \otimes_{\mathbb{C}[W]} (X \otimes \wedge^i V)$, it suffices to show

$$(3.6) \quad d_i(t_w \otimes (x \otimes v_1 \wedge \dots \wedge v_i)) = d_i(1 \otimes (t_w.x \otimes w(v_1) \wedge \dots \wedge w(v_i))).$$

For simplicity, set

$$\begin{aligned} P^w &= d_i(t_w \otimes (x \otimes v_1 \wedge \dots \wedge v_i)) \\ &= t_w \sum_{k=0}^i (-1)^{k-1} [v_i \otimes (x \otimes v_1 \wedge \dots \wedge \hat{v}_k \wedge \dots \wedge v_i) - 1 \otimes (v_k.x \otimes v_1 \wedge \dots \wedge \hat{v}_k \wedge \dots \wedge v_i)] \end{aligned}$$

and

$$\begin{aligned} P_w &= d_i(1 \otimes (t_w.x \otimes w(v_1) \wedge \dots \wedge w(v_i))) \\ &= \sum_{i=0}^k (-1)^{k-1} w(v_k) \otimes (t_w.x \otimes w(v_1) \wedge \dots \wedge \widehat{w(v_k)} \wedge \dots \wedge w(v_i)) \\ &\quad - \sum_{i=0}^k (-1)^{k-1} \otimes (w(v_i).t_w.x \otimes w(v_1) \wedge \dots \wedge \widehat{w(v_k)} \wedge \dots \wedge w(v_i)) \end{aligned}$$

To show equation (3.6), it is equivalent to show $P^w = P_w$. Regard $\mathbb{C}[W]$ as a natural subalgebra of \mathbb{H} . By using the fact that $t_w v - w(v)t_w \in \mathbb{C}[W]$ for $w \in W$, $P^w - P_w$ is an element of the form $1 \otimes u$ for some $u \in X \otimes \wedge^i V$. Thus it suffices to show that $u = 0$. To this end, by the induction hypothesis, d_{i-1} is well-defined and then a direct computation (from the original expressions of P^w and P_w) shows that $d_{i-1}(P^w - P_w) = 0$ and hence $d_{i-1}(1 \otimes u) = 0$. The statement now follows from the fact that the union of

$$\{e_{r_k} \otimes (x_{r_1, \dots, r_{i-1}}) \otimes e_{r_1} \wedge \dots \wedge \widehat{e}_{r_k} \wedge \dots \wedge e_{r_{i-1}}\}_{1 \leq r_1 < \dots < r_k < \dots < r_{i-1} \leq n}$$

and

$$\{1 \otimes e_{r_k} \cdot (x_{r_1, \dots, r_{i-1}}) \otimes e_{r_1} \wedge \dots \wedge \widehat{e}_{r_k} \wedge \dots \wedge e_{r_{i-1}}\}_{1 \leq r_1 < \dots < r_k < \dots < r_{i-1} \leq n}$$

forms a linearly independent set. Here $x_{r_1, \dots, r_k} \in X$ and e_1, \dots, e_n is a fixed basis of V .

Verifying $d^2 = 0$ is straightforward.

□

Corollary 3.2. (1) *For any \mathbb{H} -module X , the complex (3.3) forms a projective resolution for X .*

(2) *The homological dimension of \mathbb{H} is $\dim V$.*

Proof. For (1), from Proposition 3.1, we only have to show the exactness. This can be proven by an argument which imposes a grading on \mathbb{H} and uses a long exact sequence (see for example [20, Section 5.3.8]).

We now prove (2). By (1), the homological dimension of \mathbb{H} is less than or equal to $\dim V$. We now show the homological dimension attains the upper bound. Let $\gamma \in V^\vee$ be a regular element and let v_γ be a vector with weight $\gamma \in V^\vee$. Define $X = \text{Ind}_{S(V)}^{\mathbb{H}} \mathbb{C}v_\gamma$. By Frobenius reciprocity and using γ is regular, $\text{Ext}_{\mathbb{H}}^i(X, X) = \text{Ext}_{S(V)}^i(\mathbb{C}v_\gamma, \mathbb{C}v_\gamma) \neq 0$ for all $i \leq \dim V$. This shows the homological dimension has to be $\dim V$.

□

3.2. Alternate form of the Koszul-type resolution. In this section, we give another form of the differential map d_i , which involves the terms \tilde{v} (defined in (2.1)). There are some advantages for computations in later sections.

We consider the maps $\tilde{d}_i : \mathbb{H} \otimes_{\mathbb{C}[W]} (\text{Res}_W X \otimes \wedge^i V) \rightarrow \mathbb{H} \otimes_{\mathbb{C}[W]} (\text{Res}_W X \otimes \wedge^{i-1} V)$ as follows:

$$\tilde{d}_i(h \otimes (x \otimes v_1 \wedge \dots \wedge v_i)) = \sum_{j=0}^i (-1)^j (h \tilde{v}_j \otimes x \otimes v_1 \wedge \dots \wedge \widehat{v}_j \wedge \dots \wedge v_i - h \otimes \tilde{v}_j \cdot x \otimes v_1 \wedge \dots \wedge \widehat{v}_j \wedge \dots \wedge v_i).$$

We show that this definition coincides with the one in the previous subsection:

Proposition 3.3. $\tilde{d}_i = d_i$.

Proof. Recall that for $v_i \in V$,

$$\tilde{v}_i = v_i - \sum_{\alpha \in R^+} k_\alpha \langle v_i, \alpha^\vee \rangle t_{s_\alpha}.$$

Then

$$\begin{aligned}
& \tilde{v}_r \otimes (x \otimes v_1 \wedge \dots \wedge \widehat{v}_r \wedge \dots \wedge v_k) - 1 \otimes (\tilde{v}_r.x \otimes v_1 \wedge \dots \wedge \widehat{v}_r \wedge \dots \wedge v_k) \\
= & v_r \otimes (x \otimes v_1 \wedge \dots \wedge \widehat{v}_r \wedge \dots \wedge v_k) - 1 \otimes (v_r.x \otimes v_1 \wedge \dots \wedge \widehat{v}_r \wedge \dots \wedge v_k) \\
& - \sum_{\alpha \in R^+} k_\alpha \langle v_r, \alpha^\vee \rangle \otimes (t_{s_\alpha}.x) \otimes s_\alpha(v_1) \wedge \dots \wedge s_\alpha(\widehat{v}_r) \wedge \dots \wedge s_\alpha(v_k) \\
& + \sum_{\alpha \in R^+} k_\alpha \langle v_r, \alpha^\vee \rangle \otimes (t_{s_\alpha}.x) \otimes v_1 \wedge \dots \wedge \widehat{v}_r \wedge \dots \wedge v_k \\
= & v_r \otimes (x \otimes v_1 \wedge \dots \wedge \widehat{v}_r \wedge \dots \wedge v_k) - 1 \otimes (v_r.x \otimes v_1 \wedge \dots \wedge \widehat{v}_r \wedge \dots \wedge v_k) \\
& - (-1)^p \sum_{\alpha \in R^+} \sum_{p < r} k_\alpha \langle v_r, \alpha^\vee \rangle \langle v_p, \alpha^\vee \rangle \otimes (t_{s_\alpha}.x) \otimes \alpha \wedge s_\alpha(v_1) \wedge \dots \wedge s_\alpha(\widehat{v}_p) \wedge \dots \wedge s_\alpha(\widehat{v}_r) \wedge \dots \wedge s_\alpha(v_k) \\
& - (-1)^{p-1} \sum_{\alpha \in R^+} \sum_{r < p} k_\alpha \langle v_r, \alpha^\vee \rangle \langle v_p, \alpha^\vee \rangle \otimes (t_{s_\alpha}.x) \otimes \alpha \wedge s_\alpha(v_1) \wedge \dots \wedge s_\alpha(\widehat{v}_r) \wedge \dots \wedge s_\alpha(\widehat{v}_p) \wedge \dots \wedge s_\alpha(v_k)
\end{aligned}$$

With the expression above, some standard computations can verify $\tilde{d}_i = d_i$.

□

3.3. Euler-Poincaré pairing. We define the Euler-Poincaré pairing as:

$$\text{EP}_{\mathbb{H}}(X, Y) = \sum_i (-1)^i \dim \text{Ext}_{\mathbb{H}}^i(X, Y),$$

where the Ext groups are defined in the category of \mathbb{H} -modules. This pairing can be realized as an inner product on a certain elliptic space for \mathbb{H} -modules analogue to the one in p -adic reductive groups in the sense of Schneider-Stuhler [31].

The elliptic pairing $\langle, \rangle_W^{\text{ellip}, V}$ on W -representations U and U' is defined as

$$\langle U, U' \rangle_W^{\text{ellip}, V} = \frac{1}{|W|} \sum_{w \in W} \text{tr}_U(w) \overline{\text{tr}_{U'}(w)} \det_V(1 - w).$$

Proposition 3.4. *For any finite-dimensional \mathbb{H} -modules X and Y ,*

$$\text{EP}_{\mathbb{H}}(X, Y) = \langle \text{Res}_W(X), \text{Res}_W(Y) \rangle_W^{\text{ellip}, V}.$$

In particular, the Euler-Poincaré pairing depends only on the W -module structure of X and Y .

Proof.

$$\begin{aligned}
\mathrm{EP}_{\mathbb{H}}(X, Y) &= \sum_i (-1)^i \dim \mathrm{Ext}_{\mathbb{H}}^i(X, Y) \\
&= \sum_i (-1)^i (\ker d_i^* - \mathrm{im} d_{i-1}^*) \\
&= \sum_i (-1)^i \dim \mathrm{Hom}_{\mathbb{H}}(\mathbb{H} \otimes_{\mathbb{C}[W]} (\mathrm{Res}_W(X) \otimes \wedge^i V), Y) \quad (\text{by Corollary 3.2}) \\
&= \sum_i (-1)^i \dim \mathrm{Hom}_{\mathbb{C}[W-]}(\mathrm{Res}_W(X) \otimes \wedge^i V, \mathrm{Res}_W(Y)) \quad (\text{by Frobenius reciprocity}) \\
&= \sum_{w \in W} \mathrm{tr}_{\mathrm{Res}_W X}(w) \overline{\mathrm{tr}_{\mathrm{Res}_W Y}(w)} \mathrm{tr}_{\wedge^{\pm} V}(w) \\
&= \langle \mathrm{Res}_W(X), \mathrm{Res}_W(Y) \rangle_W^{\mathrm{ellip}, V}
\end{aligned}$$

Here $\wedge^{\pm} V = \bigoplus_{i \in \mathbb{Z}} (-1)^i \wedge^i V$ as a virtual representation. The last equality follows from $\mathrm{tr}_{\wedge^i V}(w) = \det(1 - w)$ and the definition.

□

4. TWISTED EULER-POINCARÉ PAIRING

Recall that θ is defined in Section 2.5. For any $\mathbb{H} \rtimes \langle \theta \rangle$ -module X , denote $\mathrm{Res}_W X$ to be the restriction of X to a $\mathbb{C}[W]$ -algebra module (Definition 2.1 (1)). The notion $\mathrm{Res}_{W \rtimes \langle \theta \rangle}$ is similarly defined.

4.1. θ -twisted Euler-Poincaré pairing. Let X and Y be $\mathbb{H} \rtimes \langle \theta \rangle$ -modules. The differential map d_i induces a map from $\mathrm{Hom}_{\mathbb{H}}(\mathbb{H} \otimes_{\mathbb{C}[W]} (\mathrm{Res}_W X \otimes \wedge^i V), Y)$ to $\mathrm{Hom}_{\mathbb{H}}(\mathbb{H} \otimes_{\mathbb{C}[W]} (\mathrm{Res}_W X \otimes \wedge^{i+1} V), Y)$. Then by the Frobenius reciprocity, the differential map also induces a map, denoted d^* from $\mathrm{Hom}_{\mathbb{C}[W]}(\mathrm{Res}_W X \otimes \wedge^i V, \mathrm{Res}_W Y)$ to $\mathrm{Hom}_{\mathbb{C}[W]}(\mathrm{Res}_W X \otimes \wedge^{i+1} V, \mathrm{Res}_W Y)$ as follows:

$$\begin{aligned}
(4.7) \quad & d_{i+1}^*(\psi)(x \otimes v_1 \wedge \dots \wedge v_{i+1}) \\
(4.8) \quad &= \sum_{j=0}^{i+1} (-1)^j v_j \cdot \psi(x \otimes v_1 \wedge \dots \wedge \widehat{v}_j \wedge \dots \wedge v_{i+1}) - \sum_{j=0}^i (-1)^j \psi(v_j \cdot x \otimes v_1 \wedge \dots \wedge \widehat{v}_j \wedge \dots \wedge v_{i+1}),
\end{aligned}$$

Define θ^* to be the linear automorphism on $\mathrm{Hom}_{\mathbb{C}[W]}(\mathrm{Res}_W X \otimes \wedge^i V, \mathrm{Res}_W Y)$ given by

$$(4.9) \quad \theta^*(\psi)(x \otimes v_1 \wedge \dots \wedge v_i) = \theta \circ \psi(\theta(x) \otimes \theta(v_1) \wedge \dots \wedge \theta(v_i)).$$

Here θ -actions on $\mathrm{Res}_W X$ and $\mathrm{Res}_W Y$ are just the natural actions from the θ -actions on X and Y (as $\mathbb{H} \rtimes \langle \theta \rangle$ -modules), and furthermore the θ -action on v_i comes from the action of θ on the corresponding Dynkin diagram.

Lemma 4.1. $\theta^* \circ d^* = d^* \circ \theta^*$

Proof.

$$\begin{aligned}
& (\theta^* \circ d^*)(\psi)(x \otimes v_1 \wedge \dots \wedge v_k) \\
&= \theta \circ d^*(\psi)(\theta(x) \otimes \theta(v_1) \wedge \dots \wedge \theta(v_k)) \\
&= \theta \circ \psi(d(\theta(x) \otimes \theta(v_1) \wedge \dots \wedge \theta(v_k))) \\
&= \sum_i (-1)^i v_r. \theta \circ \psi(\theta(x) \otimes \theta(v_1) \wedge \dots \wedge \theta(\widehat{v}_i) \wedge \dots \wedge \theta(v_k)) \\
&\quad - \sum_i (-1)^i \theta \circ \psi(\theta(v_r). \theta(x) \otimes \theta(v_1) \wedge \dots \wedge \theta(\widehat{v}_i) \wedge \dots \wedge \theta(v_k)) \\
&= \sum_i (-1)^i v_r. \theta^*(\psi)(x \otimes v_1 \wedge \dots \wedge \widehat{v}_i \wedge \dots \wedge v_k) \\
&\quad - \sum_i (-1)^i \theta^*(\psi)(v_r. x \otimes v_1 \wedge \dots \wedge \widehat{v}_i \wedge \dots \wedge v_k) \\
&= (d^* \circ \theta^*)(\psi)(x \otimes v_1 \wedge \dots \wedge v_k)
\end{aligned}$$

□

By Lemma 4.1, θ^* induces an action, still denoted θ^* on $\text{Ext}_{\mathbb{H}}^i(X, X)$. We can then define the θ -twisted Euler-Poincaré pairing $\text{EP}_{\mathbb{H}}^{\theta}$ as follows:

Definition 4.2. For $\mathbb{H} \rtimes \langle \theta \rangle$ -modules X and Y , define

$$\text{EP}_{\mathbb{H}}^{\theta}(X, Y) = \sum_i (-1)^i \text{trace}(\theta^* : \text{Ext}_{\mathbb{H}}^i(X, Y) \rightarrow \text{Ext}_{\mathbb{H}}^i(X, Y)).$$

Here we also regard X and Y to be \mathbb{H} -modules equipped with the θ -action.

We remark that this definition also makes sense for θ to be any automorphism of \mathbb{H} . However, when we prove Theorem 4.11 later, we essentially require θ to arise from w_0 in (2.2).

4.2. θ -twisted elliptic pairing on Weyl groups. We review the θ -twisted elliptic representation theory of Weyl groups in [12].

Definition 4.3. An element $w \in W$ is said to be θ -elliptic if $\det_V(1 - w\theta) \neq 0$. A θ -twisted conjugacy class is the set $\{ww_1\theta(w)^{-1} : w_1 \in W\}$ for some $w \in W$. A θ -twisted conjugacy class is said to be *elliptic* if it contains an θ -elliptic element.

Define

$$(4.10) \quad \mathcal{J}^{\theta} = \{J \subsetneq \Delta : \theta(J) = J\}.$$

Lemma 4.4. (1) *If $w \in W$ is not a θ -elliptic element, then w is θ -conjugate to an element in W_J for some $J \in \mathcal{J}^{\theta}$.*

(2) *Let $J \in \mathcal{J}^{\theta}$. If $w \in W_J$, then there exists a non-zero $\gamma \in V$ such that $w\theta(\gamma) = \gamma$.*

Proof. We first prove (1). Suppose w is not θ -elliptic element. Then there exists $\gamma \in V$ such that $w\theta(\gamma) = \gamma$. We may choose $w_1 \in W$ such that $w_1(\gamma)$ lies in the fundamental chamber. Let $\gamma' = w_1(\gamma)$. Then the stabilizer for γ' is W_J for some $J \subset \Delta$. Since γ' is in the fundamental chamber, $\theta(\gamma')$ is also in the fundamental chamber. The fact that

$w_1 w \theta(w_1^{-1}) \theta(\gamma') = \gamma'$ with standard theory for root systems (see for example [21, Theorem 1.12(a)]) forces $\theta(\gamma') = \gamma'$. We have $w_1 w \theta(w_1^{-1}) \in W_J$. It remains to show that $J \in \mathcal{J}^\theta$. For w with $w(\gamma') = \gamma'$, we also have $\theta(w)(\gamma') = \theta(w) \theta(\gamma') = \theta(w(\gamma')) = \gamma'$. Hence $\theta(J) = J$ and so $J \in \mathcal{J}^\theta$ as desired.

For (2), choose $\gamma \in V_J^\perp$. Then $\theta(\gamma) = \gamma$ and so $w\theta(\gamma) = \gamma$ for any $\gamma \in W_J$.

□

Definition 4.5. [12] For any $W \rtimes \langle \theta \rangle$ -representation U and U' , the θ -twisted elliptic pairing on U and U' is defined as:

$$\langle U, U' \rangle_W^{\theta\text{-ellip}, V} = \frac{1}{|W|} \sum_{w \in W} \text{tr}_U(w\theta) \overline{\text{tr}_{U'}(w\theta)} \det_V(1 - w\theta).$$

Since $w_0\theta = -\text{Id}_V$ on V , it is equivalent that

$$\langle U, U' \rangle_W^{\theta\text{-ellip}, V} = \frac{1}{|W|} \sum_{w \in W} \text{tr}_{U^+ - U^-}(ww_0) \overline{\text{tr}_{U'^+ - U'^-}(ww_0)} \det_V(1 + ww_0),$$

where U^+ and U^- (resp. U'^+ and U'^-) are the $+1$ and -1 -eigenspaces of $w_0\theta$ of U (resp. U'), and $U^+ - U^-$ and $U'^+ - U'^-$ are regarded as virtual representations of W .

Let $R(W \rtimes \langle \theta \rangle)$ be the virtual representation ring of $W \rtimes \langle \theta \rangle$. Since θ is an inner automorphism on W , $\text{Res}_W U$ is an irreducible W -representation for any irreducible $W \rtimes \langle \theta \rangle$ representation U . Then there exists a unique $W \rtimes \langle \theta \rangle$ representation denoted \overline{U} such that U and \overline{U} are isomorphic as W -representation but non-isomorphic as $W \rtimes \langle \theta \rangle$ -representation. Let R' be the space spanned by $U \oplus \overline{U}$ for all $U \in \text{Irr}(W \rtimes \langle \theta \rangle)$. Let

$$\overline{R}_W = R(W \rtimes \langle \theta \rangle) / R'.$$

Note that \overline{R}_W is isomorphic to $R(W)$ as vector spaces, but there is no canonical isomorphism between them. Note that R' is in the radical of $\langle \cdot, \cdot \rangle_W^{\theta\text{-ellip}, V}$ and so $\langle \cdot, \cdot \rangle_W^{\theta\text{-ellip}, V}$ descends to \overline{R}_W . A natural question is to describe $\text{rad} \langle \cdot, \cdot \rangle_W^{\theta\text{-ellip}, V}$ and is answered in Proposition 4.7.

Lemma 4.6. Let $U \in R(W \rtimes \langle \theta \rangle)$. Let $J \in \mathcal{J}^\theta$ and let $U' \in R(W_J \rtimes \langle \theta \rangle)$. If

$$\sum_{w \in W} \text{tr}_U(w\theta) \overline{\text{tr}_{\text{Ind}_{W_J}^W U'}(w\theta)} = 0,$$

then

$$\sum_{w \in W_J} \text{tr}_U(w\theta) \overline{\text{tr}_{U'}(w\theta)} = 0.$$

Proof. This follows from the following:

$$\begin{aligned} 0 &= \sum_{w \in W} \text{tr}_U(w\theta) \overline{\text{tr}_{\text{Ind}_{W_J \rtimes \langle \theta \rangle}^W U'}(w\theta)} \\ &= 2|W| \langle U, \text{Ind}_{W_J \rtimes \langle \theta \rangle}^W U' \rangle_{W \rtimes \langle \theta \rangle} - |W| \langle U, \text{Ind}_{W_J}^W U' \rangle_W \\ &= 2|W| \langle \text{Res}_{W_J \rtimes \langle \theta \rangle} U, U' \rangle_{W_J \rtimes \langle \theta \rangle} - |W| \langle \text{Res}_{W_J} U, U' \rangle_{W_J} \\ &= \frac{|W|}{|W_J|} \sum_{w \in W_J} \text{tr}_U(w\theta) \overline{\text{tr}_{U'}(w\theta)} \end{aligned}$$

Here \langle, \rangle_W and \langle, \rangle_{W_J} denotes the standard inner form on W -representations and W_J -representations respectively.

□

Proposition 4.7. (1) *The radical of $\langle, \rangle_W^{\theta\text{-ellip}, V}$ on \overline{R}_W is the image of*

$$\bigoplus_{J \in \mathcal{J}^\theta} \text{Ind}_{W_J \rtimes \langle \theta \rangle}^{W \rtimes \langle \theta \rangle} R(W_J \rtimes \langle \theta \rangle).$$

(2) *The dimension of the quotient space $\overline{R}_W / \text{rad} \langle, \rangle_W^{\theta\text{-ellip}, V}$ is equal to the number of elliptic θ -twisted conjugacy classes.*

Proof. We first prove (1). The proof follows the one in [25, Proposition 2.2.2]. Let $U \in \text{Ind}_{W_J \rtimes \langle \theta \rangle}^{W \rtimes \langle \theta \rangle} R(W_J \rtimes \langle \theta \rangle)$ for some $J \in \mathcal{L}^\theta$. Then $\chi_U(w\theta)$ vanishes for all w that is not θ -twisted conjugate to an element in W_J . Then by Lemma 4.4 (2), $\bigoplus_{J \in \mathcal{L}^\theta} \text{Ind}_{W_J \rtimes \langle \theta \rangle}^{W \rtimes \langle \theta \rangle} R(W_J \rtimes \langle \theta \rangle)$ is a subset of the radical of $\langle, \rangle_W^{\theta\text{-ellip}, V}$.

We now prove the converse direction. We pick a virtual representation $U \in \text{rad} \langle, \rangle_W^{\theta\text{-ellip}, V}$ such that $\langle U, \text{Ind}_{W_J}^W U' \rangle_{W \rtimes \langle \theta \rangle} = 0$ for all $J \in \mathcal{L}^\theta$ and $U' \in R(W_J \rtimes \langle \theta \rangle)$. By Lemma 4.6, $\text{tr}_U(w\theta) = 0$ for all $w \in W_J$ and all $J \in \mathcal{J}^\theta$. By Lemma 4.4, $\text{tr}_U(w\theta) = 0$ for any non-elliptic element w . This implies that $\text{tr}_U(w\theta) = \text{tr}_{U^+ - U^-}(ww_0) = 0$ for all w , where U^+ and U^- are the $+1$ and -1 eigenspaces for $w_0\theta$. Hence $U^+ = U^-$ and by definition $U \in \overline{R}_W$. Thus the orthogonal complement of the image of $\bigoplus_{J \in \mathcal{L}^\theta} \text{Ind}_{W_J}^W R(W_J \rtimes \langle \theta \rangle)$ in \overline{R}_W with respect to the pairing $\text{rad} \langle, \rangle_W^{\theta\text{-ellip}, V}$ is exactly zero. This proves (1).

For (2), it follows from Definition 4.5 and the fact that $\det_V(1 - w\theta)$ is non-zero if and only if w is θ -elliptic.

□

4.3. Relation between two twisted elliptic pairings.

Notation 4.8. Let X be an $\mathbb{H} \rtimes \langle \theta \rangle$ -module. Define X^\pm to be the ± 1 eigenspaces of the action of θt_{w_0} on X respectively. It is easy to see X^\pm are invariant under the action of t_w for $w \in W$ (see Lemma 4.9 below). We shall regard X^\pm as W -representations or $W \rtimes \langle \theta \rangle$ -representations. Moreover, since θt_{w_0} is diagonalizable, we also have $X = X^+ \oplus X^-$.

Lemma 4.9. *Let X be an $\mathbb{H} \rtimes \langle \theta \rangle$ -module. Then*

- (1) X^+ and X^- are $W \rtimes \langle \theta \rangle$ -invariant
- (2) Let X be an $\mathbb{H} \rtimes \langle \theta \rangle$ -module. For any $v \in V$, $\tilde{v}.X^\pm \subset X^\mp$.

Proof. (1) follows from $\theta t_{w_0} t_w = t_w t_{w_0} \theta$. (2) follows from $w_0 \theta(v) = -v$ and Lemma 2.4.

□

Lemma 4.10. *For $\mathbb{H} \rtimes \langle \theta \rangle$ -modules X and Y , define*

$$\text{Hom}_i^+ = \text{Hom}_{\mathbb{C}[W]}(X^+ \otimes \wedge^i V, Y^+) \oplus \text{Hom}_{\mathbb{C}[W]}(X^- \otimes \wedge^i V, Y^-)$$

and

$$\text{Hom}_i^- = \text{Hom}_{\mathbb{C}[W]}(X^+ \otimes \wedge^i V, Y^-) \oplus \text{Hom}_{\mathbb{C}[W]}(X^- \otimes \wedge^i V, Y^+).$$

The map d_i^* sends $\text{Hom}_i^\pm \rightarrow \text{Hom}_{i+1}^\mp$. Moreover, θ^* acts identically as $(-1)^i$ on Hom_i^+ and acts identically as $-(-1)^i$ on Hom_i^- .

Proof. The first assertion follows from Lemma 4.9 and Proposition 3.3. For the second assertion, we pick $\psi \in \text{Hom}_i^+$. Suppose $x \in X^+$ and $v_1, \dots, v_i \in V$. Then

$$\begin{aligned} & \theta^*(\psi)(x \otimes v_1 \wedge \dots \wedge v_i) \\ &= \theta.\psi(\theta(x) \otimes \theta(v_1) \wedge \dots \wedge \theta(v_i)) \\ &= t_{w_0} \theta.\psi((t_{w_0} \theta.x) \otimes w_0 \theta(v_1) \wedge \dots \wedge w_0 \theta(v_i)) \\ &= (-1)^i t_{w_0} \theta.\psi(x \otimes v_1 \wedge \dots \wedge v_i) \\ &= (-1)^i \psi(x \otimes v_1 \wedge \dots \wedge v_i) \end{aligned}$$

The forth equality follows from $w_0 \theta(v) = -v$, $t_{w_0} \theta.x = x$, and the last equality follows from $\text{im } \psi \in Y^+$. Other cases are similar.

□

With Hom_i^\pm defined in Lemma 4.10, we also define that

$$\text{Ext}^i(X, Y)^+ = \frac{\ker(d_i^* : \text{Hom}_i^+ \rightarrow \text{Hom}_i^-)}{\text{im}(d_i^* : \text{Hom}_i^- \rightarrow \text{Hom}_i^+)},$$

and similarly,

$$\text{Ext}^i(X, Y)^- = \frac{\ker(d_i^* : \text{Hom}_i^- \rightarrow \text{Hom}_i^+)}{\text{im}(d_i^* : \text{Hom}_i^+ \rightarrow \text{Hom}_i^-)}.$$

Note that by the projective resolution in (3.3),

$$(4.11) \quad \text{Ext}_{\mathbb{H}}^i(X, Y) = \text{Ext}^i(X, Y)^+ \oplus \text{Ext}^i(X, Y)^-.$$

Theorem 4.11. *For any finite-dimensional $\mathbb{H} \rtimes \langle \theta \rangle$ -modules X and Y with θ defined as in (2.2),*

$$\text{EP}_{\mathbb{H}}^\theta(X, Y) = \langle \text{Res}_{W \rtimes \langle \theta \rangle} X, \text{Res}_{W \rtimes \langle \theta \rangle} Y \rangle_W^{\theta-\text{ellip}, V}.$$

In particular, the θ -twisted elliptic pairing $\text{EP}_{\mathbb{H}}^\theta$ depends on the W -module structures of X and Y only.

Proof. Set $d_i^{*,+} = d_i^*|_{\text{Hom}_i^+}$ and $d_i^{*,-} = d_i^*|_{\text{Hom}_i^-}$.

$$\begin{aligned}
& \text{EP}_{\mathbb{H}}^{\theta}(X, Y) \\
&= \sum_i (-1)^i \text{trace}(\theta^* : \text{Ext}_{\mathbb{H}}^i(X, Y) \rightarrow \text{Ext}_{\mathbb{H}}^i(X, Y)) \\
&= \sum_i (-1)^i [(-1)^i \dim \text{Ext}_{\mathbb{H}}^i(X, Y)^+ - (-1)^i \dim \text{Ext}_{\mathbb{H}}^i(X, Y)^-] \quad (\text{by (4.11) and Lemma 4.10}) \\
&= \sum_i (\dim \text{Ext}_{\mathbb{H}}^i(X, Y)^+ - \dim \text{Ext}_{\mathbb{H}}^i(X, Y)^-) \\
&= \sum_i [(\dim \ker d_i^{*,+} - \dim \text{im } d_{i-1}^{*,+}) - (\dim \ker d_i^{*,-} - \dim \text{im } d_{i-1}^{*,-})] \\
&= \sum_i (\dim \ker d_i^{*,+} + \dim \text{im } d_{i-1}^{*,+}) - (\dim \ker d_i^{*,-} + \dim \text{im } d_{i-1}^{*,-}) \\
&= \sum_i (\dim \text{Hom}_i^+ - \dim \text{Hom}_i^-) \quad (\text{definition of } \text{Hom}^{\pm} \text{ in Lemma 4.10}) \\
&= \frac{1}{|W|} \sum_{w \in W} \text{tr}_{X^+ - X^-}(w) \overline{\text{tr}_{Y^+ - Y^-}(w)} \det_V(1 + w) \quad (\text{as virtual representations}) \\
&= \frac{1}{|W|} \sum_{w \in W} \text{tr}_X(w w_0 \theta) \overline{\text{tr}_Y(w w_0 \theta)} \det_V(1 - w w_0 \theta) \\
&= \langle \text{Res}_{W \rtimes \langle \theta \rangle}(X), \text{Res}_{W \rtimes \langle \theta \rangle}(Y) \rangle_W^{\theta\text{-ellip}, V}
\end{aligned}$$

The third last equality follows from the fact that $\sum_i \text{tr}_{\wedge^i V}(w) = \det_V(1 + w)$ and $w_0 \theta = -\text{Id}_V$.

□

Remark 4.12. We give an example to show that Theorem 4.11 is not true in general if θ is replaced by an outer automorphism on W . Let R be of type $A_1 \times A_1$. Let θ' be the Dynkin diagram automorphism interchanging two factors of A_1 . Let \mathbb{H} be the graded Hecke algebra of type $A_1 \times A_1$. Note that $\langle \cdot, \cdot \rangle_W^{\theta'\text{-ellip}, V} \equiv 0$ as $\text{tr}(w\theta') = 0$ for all $w \in W$. Here $W = S_2 \times S_2$ and $V = \mathbb{C} \oplus \mathbb{C}$. However, we may choose an \mathbb{H} -module X (e.g. the exterior tensor product of Steinberg modules) such that $\text{EP}_{\mathbb{H}}^{\theta'}(X, X) \neq 0$.

We give an interpretation of θ -twisted Euler-Poincaré pairing with the Euler-Poincaré pairing of $\mathbb{H} \rtimes \langle \theta \rangle$ -modules. Define $\text{EP}_{\mathbb{H} \rtimes \langle \theta \rangle}(X, Y) = \sum_i (-1)^i \dim \text{Ext}_{\mathbb{H} \rtimes \langle \theta \rangle}^i(X, Y)$, where $\text{Ext}_{\mathbb{H} \rtimes \langle \theta \rangle}^i$ is taken in the category of $\mathbb{H} \rtimes \langle \theta \rangle$ -modules.

Corollary 4.13. *For any finite-dimensional $\mathbb{H} \rtimes \langle \theta \rangle$ -modules X and Y ,*

$$\dim \text{Ext}_{\mathbb{H} \rtimes \langle \theta \rangle}^i(X, Y) = \frac{1}{2} \dim \text{Ext}_{\mathbb{H}}^i(X, Y) + \frac{1}{2} \text{trace}(\theta^* : \text{Ext}_{\mathbb{H}}^i(X, Y) \rightarrow \text{Ext}_{\mathbb{H}}^i(X, Y)),$$

and

$$\text{EP}_{\mathbb{H} \rtimes \langle \theta \rangle}(X, Y) = \frac{1}{2} \text{EP}_{\mathbb{H}}(X, Y) + \frac{1}{2} \text{EP}_{\mathbb{H}}^{\theta}(X, Y).$$

Proof. Note that

$$\text{Hom}_{\mathbb{C}[W] \rtimes \langle \theta \rangle}(\text{Res}_{W \rtimes \langle \theta \rangle} X \otimes \wedge^i V, \text{Res}_{W \rtimes \langle \theta \rangle} Y) \cong \begin{cases} \text{Hom}_i^+ & \text{if } i \text{ is even} \\ \text{Hom}_i^- & \text{if } i \text{ is odd} \end{cases}$$

Then by using a Koszul type resolution as in (3.3), one could see that

$$\mathrm{Ext}_{\mathbb{H} \rtimes \langle \theta \rangle}^i(X, Y) = \begin{cases} \mathrm{Ext}_i^+ & \text{if } i \text{ is even} \\ \mathrm{Ext}_i^- & \text{if } i \text{ is odd} \end{cases}$$

By Lemma 4.10, the latter expression above is equal to

$$\frac{1}{2} \dim \mathrm{Ext}_{\mathbb{H}}^i(X, Y) + \frac{1}{2} \mathrm{trace}(\theta^* : \mathrm{Ext}_{\mathbb{H}}^i(X, Y) \rightarrow \mathrm{Ext}_{\mathbb{H}}^i(X, Y)).$$

It follows from the proof of Proposition 3.4 that

$$\begin{aligned} & \mathrm{Ext}_{\mathbb{H} \rtimes \langle \theta \rangle}(X, Y) \\ &= \frac{1}{2|W|} \sum_{w \in W} \mathrm{tr}_X(w) \overline{\mathrm{tr}_Y(w)} \det_V(1 - w) + \frac{1}{2|W|} \sum_{w \in W} \mathrm{tr}_X(w\theta) \overline{\mathrm{tr}_Y(w\theta)} \det_V(1 - w\theta) \\ &= \frac{1}{2} \langle \mathrm{Res}_W(X), \mathrm{Res}_W(Y) \rangle_W^{\mathrm{ellip}, V} + \frac{1}{2} \langle \mathrm{Res}_{W \rtimes \langle \theta \rangle}(X), \mathrm{Res}_{W \rtimes \langle \theta \rangle}(Y) \rangle_W^{\theta - \mathrm{ellip}, V} \end{aligned}$$

Now the statement follows from Theorem 4.11 and Proposition 3.4. \square

Corollary 4.14. *Let X be a finite-dimensional $\mathbb{H} \rtimes \langle \theta \rangle$ -module. If $X \in \mathrm{rad}(\mathrm{EP}_{\mathbb{H}}^{\theta})$, then $X \in \mathrm{rad}(\mathrm{EP}_{\mathbb{H}})$.*

Proof. Proposition 4.7 is still valid if we replace θ by Id and replace \mathcal{J}^{θ} by \mathcal{J} , where \mathcal{J} is the set of all proper subsets of Δ . Since $\mathcal{J}^{\theta} \subseteq \mathcal{J}$, the statement follows from Proposition 4.7 and Theorem 4.11. \square

4.4. Semi-positiveness of the twisted Euler-Poincaré pairing. Let \widetilde{W} be the spin cover of W . For $\dim V$ even, let S be the irreducible basic spin representations of \widetilde{W} . For $\dim V$ odd, let S^+ and S^- be the two distinct basic spin representations of \widetilde{W} -representation and let $S = S^+ \oplus S^-$. For a more detail discussion of the spin cover \widetilde{W} or the representation S , one may refer to [4], [8] or [13]. The only property we will use in this paper is the following:

$$S \otimes S = n \wedge^{\bullet} V,$$

where $n = 1$ when $\dim V$ is even and $n = 2$ when $\dim V$ is odd. For an $\mathbb{H} \rtimes \langle \theta \rangle$ -module X , we define θ -twisted Dirac index as:

$$I^{\theta}(X) = (X^+ - X^-) \otimes S,$$

as a virtual \widetilde{W} -representation. The terminology of the θ -twisted Dirac index comes from the form of the Dirac index defined by Ciubotaru-Trapa [13] and Ciubotaru-He [12].

Proposition 4.15. *For $\mathbb{H} \rtimes \langle \theta \rangle$ -modules X_1 and X_2 ,*

$$\frac{n}{2} \mathrm{EP}_{\mathbb{H}}^{\theta}(X_1, X_2) = \langle I^{\theta}(X_1), I^{\theta}(X_2) \rangle_{\widetilde{W}},$$

where $n = 1$ if $\dim V$ is even and $n = 2$ if $\dim V$ is odd. Here $\langle \cdot, \cdot \rangle_{\widetilde{W}}$ is the standard inner product on \widetilde{W} -representations.

Proof. The proof is similar to the one in [13, Proposition 3.1].

$$\begin{aligned}
& \langle I^\theta(X_1), I^\theta(X_2) \rangle_{\widetilde{W}} \\
&= \langle (X_1^+ - X_1^-) \otimes S, (X_2^+ - X_2^-) \otimes S \rangle_{\widetilde{W}} \\
&= \langle X_1^+ - X_1^-, (X_2^+ - X_2^-) \otimes S \otimes S \rangle_{\widetilde{W}} \\
&= n \langle X_1^+ - X_1^-, (X_2^+ - X_2^-) \otimes \wedge^\bullet V \rangle_{\widetilde{W}} \\
&= \frac{n}{2} \langle X_1, X_2 \rangle_W^{\theta\text{-ellip}, V} \\
&= \frac{n}{2} \text{EP}_{\mathbb{H}}^\theta(X_1, X_2) \quad (\text{by Theorem 4.11})
\end{aligned}$$

□

Corollary 4.16. *The θ -twisted Euler-Poincaré pairing $\text{EP}_{\mathbb{H}}^\theta$ is semi-positive definite.*

4.5. Twisted elliptic space. Let $K_{\mathbb{C}}(\mathbb{H} \rtimes \langle \theta \rangle)$ be the Grothendieck group of the category of finite-dimensional \mathbb{H} -modules over \mathbb{C} . We have seen from Theorem 4.11 that $\text{EP}_{\mathbb{H}}^\theta$ does not depend on the choice of a representative of an element in $K_{\mathbb{C}}(\text{Mod}_{\text{fin}}(\mathbb{H} \rtimes \langle \theta \rangle))$. Hence we can extend $\text{EP}_{\mathbb{H}}^\theta$ to a Hermitian form, still denoted $\text{EP}_{\mathbb{H}}^\theta$ on $K_{\mathbb{C}}(\text{Mod}_{\text{fin}}(\mathbb{H} \rtimes \langle \theta \rangle))$.

For any irreducible $\mathbb{H} \rtimes \langle \theta \rangle$ -module X , there are two possibilities:

- (1) Suppose $X|_{\mathbb{H}}$ is reducible. Then $X|_{\mathbb{H}}$ is the sum of two non-isomorphic irreducible \mathbb{H} -modules, denoted X_1 and X_2 . In this case, $\theta(X_1) = X_2$ and so $\text{tr}_{\text{Res}_W X}(w\theta) = 0$ for all $w \in W$. By Theorem 4.11, X is in $\text{rad}(\text{EP}_{\mathbb{H}}^\theta)$.
- (2) Suppose $X|_{\mathbb{H}}$ is irreducible. Then there exists another $\mathbb{H} \rtimes \langle \theta \rangle$ -module, denoted \overline{X} such that \overline{X} and X are isomorphic as \mathbb{H} -modules, but non-isomorphic as $\mathbb{H} \rtimes \langle \theta \rangle$ -modules. More precisely, let π_X and $\pi_{\overline{X}}$ be the maps defining the action of $\mathbb{H} \rtimes \langle \theta \rangle$ on X and \overline{X} respectively. Those maps satisfy $\pi_{\overline{X}}(\theta) = -\pi_X(\theta)$. This implies $X \oplus \overline{X}$ lies in $\text{rad}(\text{EP}_{\mathbb{H}}^\theta)$ by Theorem 4.11.

Let K^1 be the subspace of $\text{rad}(\text{EP}_{\mathbb{H}}^\theta)$ spanned by all X with X in case (1) (i.e. $X|_{\mathbb{H}}$ being reducible). Let K_2 be the subspace of $\text{rad}(\text{EP}_{\mathbb{H}}^\theta)$ spanned by all $X \oplus \overline{X}$ for all X in case (2) (i.e. $X|_{\mathbb{H}}$ being irreducible). We define the space

$$K_{\mathbb{H}}^\theta = K_0(\text{Mod}_{\text{fin}}(\mathbb{H} \rtimes \langle \theta \rangle)) / (K^1 \oplus K^2).$$

Note that the image of all irreducible \mathbb{H} -modules X with the property that $X^\theta \cong X$ forms a basis on $K_{\mathbb{H}}^\theta$.

Since K^1 and K^2 are in the radical of $\text{EP}_{\mathbb{H}}^\theta$, $\text{EP}_{\mathbb{H}}^\theta$ descends to $K_{\mathbb{H}}^\theta$. We define the twisted elliptic space to be:

$$\text{Ell}_{\mathbb{H}}^\theta = K_{\mathbb{H}}^\theta / \text{rad}(\text{EP}_{\mathbb{H}}^\theta).$$

Corollary 4.17. *The space $\text{Ell}_{\mathbb{H}}^\theta$ is equipped with $\text{EP}_{\mathbb{H}}^\theta$ as an inner product.*

Proof. The assertion follows from Corollary 4.16 and our construction of $\text{Ell}_{\mathbb{H}}^\theta$.

□

The space $\text{rad}(\text{EP}_{\mathbb{H}}^\theta)$ will be discussed more in Section 6.

5. θ^* -ACTION ON EXT-GROUPS OF RIGID MODULES

5.1. Ext-groups of rigid modules. Recall that tempered modules are defined in Definition 2.14. The notion for a parabolically induced module is given in Notation 2.16.

The rigid modules are parabolically induced and tempered modules with a special kind of induced data described in the following definition.

Definition 5.1. Let \mathcal{J}_{rig} be the collection of subsets J of Δ such that

$$(5.12) \quad \text{card}(\{w \in W : w(J) = J\}) = 1.$$

Let Ξ_{rig} be the collection of $(J, U, \nu) \in \Xi$ such that $J \in \mathcal{J}_{\text{rig}}$. An \mathbb{H} -module X is said to be a *rigid module* if $X = X(J, U)$ for some $(J, U, 0) \in \Xi_{\text{rig}}$. In particular, a rigid module is a tempered and parabolically induced module.

Remark 5.2. We give two remarks on our definition of rigid modules:

- (1) The term "rigid" refers to the special choice of J in the induction datum for a rigid module. Such induction datum provides nice structures such as discussed Lemma 5.6 and Lemma 5.7 below for computing the Ext-groups and θ^* -action without introducing more tools.
- (2) The essential algebraic structure we need in our later computations is described in Lemma 5.6. The way we formulate the definition is easier to connect to the tempered modules in Section 6. As mentioned in the introduction, rigid modules provide examples of solvable tempered modules, which will be discussed in the Section 6.

Remark 5.3. For the case $\theta = \text{Id}_V$ (i.e. non-simply laced types, E_7 , E_8 and D_n (n even)), $w_0 w_J(J) = J$ for any J and hence only Δ can satisfy (5.12). For the case that $\theta \neq \text{Id}_V$ (i.e. A_n , D_n (n odd) and E_6), $J \subset \Delta$ satisfies (5.12) in Definition 5.1 if and only if $J = \Delta$ or J is in one of the following case:

- (1) in type A_n and if we identify subsets of Δ (up to conjugation in W) with partitions of n , J corresponds to a partition of distinct parts, or equivalently J is of type $A_{m_1} \times \dots \times A_{m_k}$ with all m_i mutually distinct and $m_1 + \dots + m_k = n - k$ or $n - k + 1$;
- (2) D_n (n odd) and J is of type A_{n-1} ;
- (3) E_6 and J is of type D_5 or $A_4 \times A_1$.

From the classification, it is easy to see that all rigid modules satisfy (1) in the three conditions of Proposition 2.11.

Lemma 5.4. *Let J be a subset of Δ . If $J \in \mathcal{J}_{\text{rig}}$, then there does not exist $J' \in \mathcal{J}^\theta$ such that $w(J) \subset J' \subsetneq \Delta$ for some $w \in W$. Here \mathcal{J}_{rig} is defined in (4.10).*

Proof. This is an easy case-by-case checking with the use of Remark 5.3.

□

To analyze the structure of rigid modules, we need the following result in [5] about weight spaces:

Proposition 5.5. [5] *Let $(J, U, \nu) \in \Xi$ and $X = X(J, U, \nu)$. Then the weights of the \mathbb{H} -module X are*

$$(5.13) \quad \{w(\gamma) \in V^\vee : w \in W^J, \gamma \text{ is a weight of } U \otimes \mathbb{C}_\nu\},$$

where W^J is the set of minimal representative in the coset W/W_J . Moreover, the multiplicity of a weight in X coincides with the number of times of the weight appearing in the set (5.13).

Proof. We sketch the proof here. Recall that $\text{Ind}_{\mathbb{H}_J}^{\mathbb{H}} U = \mathbb{H} \otimes_{\mathbb{H}_J} U$. By definition,

$$\{t_w \otimes u \in w \in W^J \text{ and } u \in U\}$$

spans the space $\text{Ind}_{\mathbb{H}_J}^{\mathbb{H}} U$. Then we set

$$F_i = \text{span} \{t_w \otimes u : w \in W^J \text{ and } l(w) \leq i \text{ and } u \in U\}.$$

Then the graded space $Gr(X) := \bigoplus_{i \in \mathbb{Z}} F_i / F_{i-1}$ have the same weight spaces as X . This proves the proposition. \square

Lemma 5.6. *Let (J, U) and (J, U') be in Ξ_{rig} . Then there exists \mathbb{H}_J -modules Y and Y' such that $\text{Res}_{\mathbb{H}_J} X(J, U) = U \oplus Y$ and $\text{Res}_{\mathbb{H}_J} X(J, U') = U' \oplus Y'$ as \mathbb{H}_J -modules, and*

$$\text{Ext}_{\mathbb{H}_J}^i(U, Y') = 0 \quad \text{for all integers } i.$$

Proof. By considering the central characters of the \mathbb{H}_J -submodules of X and using Theorem 2.19, X can be written as $X = U_1 \oplus Y$, where Y is the maximal \mathbb{H}_J -submodule of X with all weights of U_1 in V_J , and Y is the maximal \mathbb{H}_J -submodule with all weights γ of Y not in V_J .

We now show that $U_1 = U$. According to Proposition 5.5, for any weight γ of Y_1 , $\gamma = w(\sum a_{\alpha^\vee} \alpha^\vee)$, where $a_{\alpha^\vee} < 0$, $w \in W^J$ and α^\vee runs for all the simple coroots in R_J^\vee . Since $w(\alpha^\vee) > 0$ for all simple coroots in R_J^\vee and $\gamma \in V_J^\vee$, this forces $w(\alpha^\vee) \in R_J^\vee$. Combining the conditions that $w(\alpha^\vee) > 0$ and $w(\alpha^\vee) \in R_J^\vee$, we have w sends all the positive coroots in R_J^\vee to the positive coroots in R_J^\vee . Hence, w permutes the simple coroots in R_J^\vee and so $w(J) = J$. Now the condition that X is rigid implies that $w = 1$. By counting the multiplicity of weights, we have $U_1 = U$ as desired.

Similarly, we get the decomposition $X' = U' \oplus Y'$ for Y' similarly defined as Y . By considering the central characters of U and Y' as \mathbb{H}_J -modules and using Theorem 2.19, we have the last assertion about Ext-groups in the statement. \square

Lemma 5.7. *Let $(J, U, 0) \in \Xi_{\text{rig}}$. Then the rigid module $X(J, U)$ is irreducible.*

Proof. Set $X = X(J, U)$. By Proposition 2.17, X is isomorphic to the direct sum of irreducible \mathbb{H} -modules. Now by Frobenius reciprocity and Lemma 5.6,

$$\text{Hom}_{\mathbb{H}}(X, X) = \text{Hom}_{\mathbb{H}_J}(U, \text{Res}_{\mathbb{H}_J} X) = \text{Hom}_{\mathbb{H}_J}(U, U) = \mathbb{C}.$$

This implies X is irreducible.

□

Form Lemma 5.6, we see that the computation of Ext-groups for a rigid module $X(J, U)$ can be reduced to compute the Ext-groups $\text{Ext}_{\mathbb{H}_J}^i(U, U)$. The study for the Ext-groups among discrete series is out of scope from our development. We need the following result from Opdam-Solleveld for Proposition 5.11 later:

Theorem 5.8. [28, Theorem 3.8] *Let U and U' be discrete series of $\overline{\mathbb{H}}_J$. Then*

$$\text{Ext}_{\mathbb{H}_J}^i(U, U') = \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ and } U \cong U' \\ 0 & \text{otherwise} \end{cases}$$

Proof. Apply the result [28, Theorem 3.8] for affine Hecke algebras. The result can be interpreted in the level of the graded affine Hecke algebra by using Lusztig's reduction theorem [24] (See the discussions in [32, Section 6]).

□

Example 5.9. We consider the Steinberg module St of \mathbb{H} , which is a one dimensional space $\mathbb{C}x$ with \mathbb{H} -action defined by:

$$t_{s_\alpha}.x = -x \quad \text{for } \alpha \in \Delta,$$

$$v.x = \rho(v)x,$$

where ρ is the half sum of all the positive coroots in R^\vee . Then $\text{Res}_W \text{St} = \text{sgn}$, the sign representation of W . By the projective resolution in Corollary 3.2 and notations in Section 4.1,

$$\text{Ext}_{\mathbb{H}}^i(\text{St}, \text{St}) = \frac{\ker d^* : \text{Hom}_W(\text{sgn} \otimes \wedge^i V, \text{sgn}) \rightarrow \text{Hom}_W(\text{sgn} \otimes \wedge^{i+1} V, \text{sgn})}{\text{im } d^* : \text{Hom}_W(\text{sgn} \otimes \wedge^{i-1} V, \text{sgn}) \rightarrow \text{Hom}_W(\text{sgn} \otimes \wedge^i V, \text{sgn})}.$$

Recall that the map d^* is determined by the \mathbb{H} -module structure of St . It is well-known that $\{\wedge^i V\}_{i=0}^{\dim V}$ are irreducible and mutually non-isomorphic W -representations. Hence

$$\text{Hom}_W(\text{sgn} \otimes \wedge^i V, \text{sgn}) = \begin{cases} \mathbb{C} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence we have $\text{Ext}_{\mathbb{H}}^i(\text{St}, \text{St}) = \mathbb{C}$ for $i = 0$ and $\text{Ext}_{\mathbb{H}}^i(\text{St}, \text{St}) = 0$ for $i > 0$ as stated in Theorem 5.8.

In order to reduce the amount of notation below, for \mathbb{H} -module X, X' , we simply write $\text{Hom}_W(X \otimes \wedge^i V, X')$ for $\text{Hom}_W(\text{Res}_W(X) \otimes \wedge^i V, \text{Res}_W(X'))$. Similar notation is also used for Hom functor for W_J -representations.

Notation 5.10. Let $J \subset \Delta$ and let U and U' be W_J -representations. In Proposition 5.11 below, we frequently regard the spaces $\text{Hom}_{W_J}(U \otimes \wedge^l V_J \otimes \wedge^{i-l} V_J^\perp, U')$ and $\text{Hom}_{W_J}(U \otimes \wedge^l V_J, U')$ as natural subspaces of $\text{Hom}_{W_J}(U \otimes \wedge^i V, U')$ and $\text{Hom}_{W_J}(U \otimes \wedge^l V, U')$ respectively. In Lemma 5.14, $\text{Hom}_{W_J}(U \otimes \wedge^i V_J^\perp, U)$ is regarded as a natural subspace of $\text{Hom}_{W_J}(U \otimes \wedge^l V, U)$.

Proposition 5.11. *Let $(J, U), (J, U') \in \Xi_{\text{rig}}$. Then*

$$\dim \text{Ext}_{\mathbb{H}}^i(X(J, U), X(J, U')) = \begin{cases} \binom{r}{i} = \frac{r!}{(r-i)!i!} & \text{if } U \cong U' \text{ and } i \leq r \\ 0 & \text{otherwise.} \end{cases}$$

where $r = \dim V - \dim V_J$.

Proof. Let $X = X(J, U)$ and $X' = X(J, U')$. By Lemma 5.6 and Frobenius reciprocity, $\text{Ext}_{\mathbb{H}}^i(X, X') = \text{Ext}_{\mathbb{H}_J}^i(U, U' \oplus Y') = \text{Ext}_{\mathbb{H}_J}^i(U, U')$, where Y' is an \mathbb{H}_J -module as in Lemma 5.6. We write $V = V_J \oplus V_J^\perp$. For notational convenience, we shall simply write U for $\text{Res}_{W_J}(U)$ below, which should not cause confusion.

We now apply the projective resolution in (3.3) on the graded Hecke algebra \mathbb{H}_J which have the root datum $(R_J, V_0, R_J^\vee, V_0^\vee)$ and use $d_{i,U}^*$ for the corresponding differential map as in (4.7) and (4.8). Note that we could decompose the space

$$(5.14) \quad \text{Hom}_{W_J}(U \otimes \wedge^i V, U') = \bigoplus_{l=0}^i \text{Hom}_{W_J}(U \otimes \wedge^l V_J \otimes \wedge^{i-l} V_J^\perp, U')$$

$$(5.15) \quad = \bigoplus_{l=0}^i a_{r,i,l} \text{Hom}_{W_J}(U \otimes \wedge^l V_J, U'),$$

where $a_{r,i,l} = C_{i-l}^r$ if $i-l \leq r$ and $a_{r,i,l} = 0$ if $i-l > r$. Under the above isomorphism, the map $d_{i,U}^*$ and can be in turn expressed as

$$\bigoplus_{l=0}^i d_{l,U}^* : \text{Hom}_{W_J}(U \otimes \wedge^l V_J, U') \rightarrow \text{Hom}_{W_J}(U \otimes \wedge^{l+1} V_J, U'),$$

where $\text{Hom}_{W_J}(U \otimes \wedge^l V_J, U')$ and $\text{Hom}_{W_J}(U \otimes \wedge^{l+1} V_J, U')$ are regarded as subspaces of $\text{Hom}_{W_J}(U \otimes \wedge^l V, U')$ and $\text{Hom}_{W_J}(U \otimes \wedge^{l+1} V, U')$ and by abuse of notation, $d_{i,U}^*$ are the maps restricted to the subspaces. Then the Ext-groups can be expressed as

$$(5.16) \quad \text{Ext}_{\mathbb{H}}^i(X, X') \\ (5.17) \quad = \bigoplus_{l=0}^i a_{r,i,l} \frac{\ker(d_{l,U}^* : \text{Hom}_{W_J}(U \otimes \wedge^l V_J, U') \rightarrow \text{Hom}_{W_J}(U \otimes \wedge^{l+1} V_J, U'))}{\text{im}(d_{l-1,U}^* : \text{Hom}_{W_J}(U \otimes \wedge^{l-1} V_J, U') \rightarrow \text{Hom}_{W_J}(U \otimes \wedge^l V_J, U'))}.$$

Then we have

$$(5.18) \quad \text{Ext}_{\mathbb{H}}^i(X, X') = \bigoplus_{l=0}^i a_{r,i,l} \text{Ext}_{\mathbb{H}_J}^l(U, U').$$

By Theorem 5.8, we obtain the statement. \square

5.2. θ^* -action on Ext-groups of rigid modules. This subsection is devoted to compute the θ -action on Ext-groups of rigid modules.

Let $(J, U, 0) \in \Xi_{\text{rig}}$. Define an $\overline{\mathbb{H}}_{\theta(J)}$ -module U^θ such that U^θ is identified with U as vector spaces and the $\overline{\mathbb{H}}_{\theta(J)}$ -module structure is determined by: for $u \in U$,

$$\pi_{U^\theta}(t_w)u = \pi_U(\theta(t_w))u, \text{ for } w \in W_{\theta(J)}$$

$$\pi_{U^\theta}(v)u = \pi_U(\theta(v))u, \text{ for } v \in V.$$

Lemma 5.12. *Let $(J, U, 0) \in \Xi_{\text{rig}}$. Then $X(\theta(J), U^\theta)$ and $X(J, U)$ are isomorphic.*

Proof. Set $X = X(J, U)$. By Corollary 2.13 and Proposition 2.17, X^θ and X are isomorphic. This implies $\text{Hom}_{\mathbb{H}_{\theta(J)}}(U^\theta \otimes \mathbb{C}_0, X) \neq 0$. Then the irreducibility of X in Lemma 5.7 and Frobenius reciprocity implies the statement. \square

By Lemma 5.12, $\mathbb{H} \otimes_{\mathbb{H}_J} U \cong \mathbb{H} \otimes_{\mathbb{H}_{\theta(J)}} U^\theta$ via a map denoted $T_{(J,U)}$. We also define another map $T_\theta : \mathbb{H} \otimes_{\mathbb{H}_J} U \rightarrow \mathbb{H} \otimes_{\mathbb{H}_{\theta(J)}} U^\theta$ given by $\theta(h) \otimes u \mapsto h \otimes u$. Then the map $T_{(J,U)}^{-1} \circ T_\theta$ defines an θ -action on $\mathbb{H} \otimes_{\mathbb{H}_J} U$ and gives an $\mathbb{H} \rtimes \langle \theta \rangle$ -structure on $\mathbb{H} \otimes_{\mathbb{H}_J} U$. Then we see that for any $x \in \mathbb{H} \otimes_{\mathbb{H}_J} U$, x can be uniquely written as the linear combination of

$$x = \sum_{w \in W^{\theta(J)}} t_w \theta(u_w),$$

for some $u_w \in U$.

Recall from Section 2.1 that for $J \subset \Delta$, w_0^J denotes the longest element in W^J .

Lemma 5.13. *Let X , U and Y be as in Lemma 5.6. Regard U and Y as subspaces of X (see the proof of Lemma 5.6). Then*

- (1) *Fix a choice of an involution θ_J on U induced from the longest element in W_J . For any non-zero vector $u \in U$, there exists a non-zero scalar a such that u can be uniquely written as*

$$\theta_J(u) = a t_{w_0^{\theta(J)}} \theta(u) + \sum_{w \in W^{\theta(J)} \setminus \{w_0^{\theta(J)}\}} t_w \theta(u_w)$$

for some $u_w \in U$. (Different choice of the θ_J action changes the sign of the scalar a).

- (2) *Y is the linear subspace of X spanned by all vectors of the form*

$$(5.19) \quad t_w \theta(u), \text{ for } u \in U \text{ and for } w \in W^{\theta(J)} \setminus \{w_0^{\theta(J)}\}.$$

Proof. We define Y' to be the subspace of X spanned by all vectors of the form $t_w \theta(u)$ for $w \in W^J$ and $u \in U$. Then there is a natural projection map $\text{pr} : U \hookrightarrow X \rightarrow X/Y'$. Note that any generalized weight vector of the form

$$t_{w_0^{\theta(J)}} \theta(u_w) + y, \text{ for } y \in Y'$$

has a weight $\theta(w_0^{\theta(J)}(\gamma)) = -w_{0,J}(\gamma)$ for some $\gamma \in V_J$. Then by the definition of non- θ -induced and using similar argument as in the proof of Lemma 5.6, any generalized weight vector of X lies in Y' does not have a weight in V_J . Hence $U \cap Y' = 0$ and by considering the dimension, the map pr is a linear isomorphism. Using the uniqueness of expression in (5.19), we have a map f from U to U such that

$$\theta_J(u) = t_{w_0^{\theta(J)}} \theta(f(u)) + y, \text{ for } y \in Y'.$$

We shall show that $f \circ \theta_J$ is an \mathbb{H}_J -module isomorphism.

We next prove that Y' is invariant under $w \in W_J$. It suffices to show that for any $w \in W_J$, $ww_0^{\theta(J)}W_{\theta(J)} = w_0^{\theta(J)}W_{\theta(J)}$ as cosets. Indeed this follows from

$$ww_0^{\theta(J)}W_{\theta(J)} = ww_0w_{0,\theta(J)}W_{\theta(J)} = w_0\theta(w)w_{0,\theta(J)}W_{\theta(J)} = w_0W_{\theta(J)}.$$

Note that we also have that Y' is invariant under the action of $S(V)$. Hence Y is an \mathbb{H}_J -module.

Now by using the uniqueness property in Lemma 5.19 with some computations, one can show that $f \circ \theta_J(t_w.u) = t_w.f \circ \theta_J(u)$ for $w \in W_J$ and $f \circ \theta_J(v.u) = v.f \circ \theta_J(u)$. This proves the claim that $f \circ \theta_J$ is an \mathbb{H}_J -module isomorphism and Hence $f = a\theta_J$ for some nonzero scalar a . This proves (1).

Note that by our description of Y in the proof of Lemma 5.6 and the fact that any generalized weight vector of Y' does not have a weight in V_J , we have $Y = Y'$.

□

Let $X = X(J, U)$ be a rigid module. Lemma 5.14(1) below shows $\text{Ext}_{\mathbb{H}}^i(X, X)$ can be identified with a subspace of $\text{Hom}_{W_J}(U \otimes \wedge^i V_J^\perp, U)$. Recall that the θ^* -action on $\text{Ext}_{\mathbb{H}}^i(X, X)$ is defined in Section 4.1. However, there is no natural way to define a corresponding action of θ^* on $\text{Hom}_{W_J}(U \otimes \wedge^i V_J^\perp, U)$ in general. Thus for $\psi \in \text{Hom}_{W_J}(U \otimes \wedge^i V_J, U)$, we define $\overline{\psi} \in \text{Hom}_W(X \otimes \wedge^i V, X)$ such that

$$\overline{\psi}((t_w.u) \otimes (v_1 \wedge \dots \wedge v_i)) = t_w\psi(u \otimes (w^{-1}(v_1) \wedge \dots \wedge w^{-1}(v_i)))$$

for any $w \in W$ and $u \in U$. Here we regard U as a natural subspace of $X \cong \mathbb{H} \otimes_{\mathbb{H}_J} U$ by sending u to $1 \otimes u$.

Lemma 5.14. *Let $X = X(J, U)$ be a rigid module. Regard U as a natural subspace of $X \cong \mathbb{H} \otimes_{\mathbb{H}_J} U$. Let*

$$d_i^* : \text{Hom}_W(X \otimes \wedge^i V, X) \rightarrow \text{Hom}_W(X \otimes \wedge^{i+1} V, X)$$

and

$$d_{i,U}^* : \text{Hom}_{W_J}(U \otimes \wedge^i V, U) \rightarrow \text{Hom}_{W_J}(U \otimes \wedge^{i+1} V, U)$$

be the differential maps for the \mathbb{H} -module X and the \mathbb{H}_J -module $U \otimes \mathbb{C}_0$ given by (4.8).

- (1) The map $\psi \mapsto \overline{\psi}$ induces an isomorphism between the complexes $\{d_{i,U}^*, \text{Hom}_{W_J}(U \otimes \wedge^i V, U)\}$ and $\{d_i^*, \text{Hom}_W(X \otimes \wedge^i V, X)\}$. The inverse map is given by the map restricting $X \otimes \wedge^i V$ to $U \otimes \wedge^i V$ (as W_J -representations).
- (2) Define $d_i^{U,*}$ to be the restriction of $d_{i,U}^*$ to the subspace $\text{Hom}_{W_J}(U \otimes \wedge^i V_J^\perp, U)$ (see notation 5.10). Then $\text{Ext}_{\mathbb{H}}^i(X, X)$ can be identified with $\ker d_i^{U,*}$.
- (3) We use the identification in (2). For any $\psi \in \text{Ext}_{\mathbb{H}}^i(X, X) \subset \text{Hom}_{W_J}(U \otimes \wedge^i V_J^\perp, U)$, ψ is the multiplication of a scalar in the following sense:
for each fixed $v_1 \wedge \dots \wedge v_i \in \wedge^i V_J^\perp$, there exists a scalar $\lambda_{v_1 \wedge \dots \wedge v_i}$ such that $\psi(u \otimes v_1 \wedge \dots \wedge v_i) = \lambda_{v_1 \wedge \dots \wedge v_i} u$ for all $u \in U$.
- (4) We use the identification in (2). For any $\psi \in \ker d_i^{U,*}$, the map $\theta^*(\overline{\psi})$ is equal to $(-1)^i \overline{\psi} + \phi$ for some $\phi \in \text{im } d_{i-1}^*$.

Proof. Express $X = U \oplus Y$ as in Lemma 5.6. Note that the natural inclusion $U \hookrightarrow \mathbb{H}_{\mathbb{H}_J} U \cong X$ coincides with the natural inclusion $U \hookrightarrow U \oplus Y \cong X$.

We consider (1). As W_J -representations, $\text{Res}_{W_J} X = \mathbb{C}[W] \otimes_{\mathbb{C}[W_J]} \text{Res}_{W_J} U$. (1) follows from the Frobenius reciprocity and the fact that $\text{Ext}_{\mathbb{H}_J}(U, Y) = 0$ in Lemma 5.6.

(2) is implicitly proved in Proposition 5.11. Indeed the expression follows from the identifications in (5.14), (5.15) and (5.17). Note that from (5.14) to (5.15), we drop $\wedge^i V_J^\perp$ because W_J acts trivially on V_J^\perp . However θ does not act trivially on V_J^\perp and so we recover V_J^\perp for the computation of θ^* -action here.

For (3), note that from the proof of Proposition of 5.11, we also have

$$\text{Ext}_{\mathbb{H}}^i(X, X) = \ker d_i^{U,*} \cong \text{Hom}_{\mathbb{H}_J}(U, U) \otimes \wedge^i V_J^\perp.$$

Then the result follows from the Schur's lemma.

We now prove (4). Pick an element $u \in U$. By Lemma 5.13, $\theta_J(u) = at_{w_0^{\theta(J)}}\theta(u) + y$ for some non-zero scalar a and for $y \in Y$.

Without loss of generality, we pick ψ as in (3). For $v_1 \wedge \dots \wedge v_i \in \wedge^i V_J^\perp$,

$$\begin{aligned} & \theta^*(\overline{\psi})(\theta_J(u) \otimes v_1 \wedge \dots \wedge v_i) \\ &= \theta^*(\overline{\psi})((at_{w_0^{\theta(J)}}\theta(u) + y) \otimes v_1 \wedge \dots \wedge v_i) \\ &= a\theta\overline{\psi}(t_{w_0^J}u \otimes \theta(v_1) \wedge \dots \wedge \theta(v_i)) + \theta^*(\overline{\psi})(y \otimes v_1 \wedge \dots \wedge v_i) \\ &= at_{w_0^{\theta(J)}}\theta\overline{\psi}(u \otimes (w_0^J)^{-1}\theta(v_1) \wedge \dots \wedge (w_0^J)^{-1}\theta(v_i)) + \theta^*(\overline{\psi})(y \otimes v_1 \wedge \dots \wedge v_i) \\ &= (-1)^i at_{w_0, \theta(J)}\lambda_{v_1 \wedge \dots \wedge v_i}\theta(u) + \theta^*(\overline{\psi})(y \otimes v_1 \wedge \dots \wedge v_i) \quad \text{by (3)} \\ &= (-1)^i \lambda_{v_1 \wedge \dots \wedge v_i}\theta_J(u) - (-1)^i \lambda_{v_1 \wedge \dots \wedge v_i}y + \theta^*(\overline{\psi})(y \otimes v_1 \wedge \dots \wedge v_i) \\ &= (-1)^i \overline{\psi}(\theta_J(u) \otimes v_1 \wedge \dots \wedge v_i) - (-1)^i \lambda_{v_1 \wedge \dots \wedge v_i}y + \theta^*(\overline{\psi})(y \otimes v_1 \wedge \dots \wedge v_i) \end{aligned}$$

We now define $\phi'(\theta_J(u) \otimes v_1 \wedge \dots \wedge v_i) = -(-1)^i \lambda_{v_1 \wedge \dots \wedge v_i}y + \theta^*(\overline{\psi})(y \otimes v_1 \wedge \dots \wedge v_i)$ if $v_1 \wedge \dots \wedge v_i \in \wedge^i V_J^\perp$ and $\phi'(\theta_J(u) \otimes v_1 \wedge \dots \wedge v_i) = 0$ otherwise. Note that $\theta^*(\overline{\psi})(y \otimes v_1 \wedge \dots \wedge v_i)$ is in Y by using Lemma 5.13 (2) and hence $\phi' \in \text{Hom}_{W_J}(U \otimes \wedge^i V, Y)$. Since $\text{Ext}_{\mathbb{H}_J}^i(U, Y) = 0$, this implies that $\phi' \in \text{im } d_{i-1}^*$ by definition. Now $\theta^*(\overline{\psi}) - (-1)^i \overline{\psi} - \phi'$ is indeed a map lying in the subspace

$$\bigoplus_{l=1}^i \text{Hom}_{W_J}(U \otimes \wedge^l V_J \otimes \wedge^{i-l} V_J^\perp, U).$$

This is again in $\text{im } d_{i-1}$ by following some computation in Proposition of 5.11 and we omit the detail. □

Theorem 5.15. *Let \mathbb{H} be the graded affine Hecke algebra associated to a crystallographic root system. Let $X = X(J, U)$ and $X' = X(J, U')$ for some $(J, U, 0), (J, U', 0) \in \Xi_{\text{rig}}$ (i.e. X and X' are rigid modules (Definition 5.1)). Then*

$$\dim \text{Ext}_{\mathbb{H}}^i(X, X') = \begin{cases} \binom{r}{i} = \frac{r!}{(r-i)!i!} & \text{if } U \cong U' \text{ and } i \leq r \\ 0 & \text{otherwise,} \end{cases}$$

where $r = \dim V - \dim V_J$. θ^* defined in (4.9) acts by the multiplication of a scalar of $(-1)^i$ on $\text{Ext}^i(X, X')$.

Proof. The first assertion is Proposition 5.11. For the second assertion, we only have to consider $U' = U$ in view of Proposition 5.11. With Lemma 5.14 (1), we rewrite

$$\text{Ext}_{\mathbb{H}}^i(X, X) = \ker(d_i^{U,*} : \text{Hom}_{W_J}(U \otimes \wedge^i V_J^\perp, U) \rightarrow \text{Hom}_{W_J}(U \otimes \wedge^{i+1} V, U)),$$

Now using Lemma 5.14 (1) and (4), we have that θ^* acts by $(-1)^i$ on $\text{Ext}_{\mathbb{H}}^i(X, X)$.

□

Remark 5.16. The author would like to thank Maarten Solleveld for pointing out [30, Theorem 5.2].

The Ext-groups for arbitrary tempered modules can be computed from a simple formula in [30, Theorem 5.2]. In particular, if $X = X(J, U)$ for some $(J, U, 0) \in \Xi$ and X is irreducible, then $\text{Ext}_{\mathbb{H}}^i(X, X) \cong \wedge^i V_J^\perp$. However, it seems not to be direct to know the θ^* -action on the Ext-groups from [30].

As a consequence of Theorem 5.15 and Corollary 4.13, we have the following result.

Corollary 5.17. *Let $X = X(J, U)$ be a rigid module of discrete series. Set $r = \dim V_J^\perp$. Then*

- (1) $\text{EP}_{\mathbb{H}}^\theta(X, X) = 2^r \neq 0$.
- (2) $\dim \text{Ext}_{\mathbb{H} \rtimes \langle \theta \rangle}^i(X, X) = \binom{r}{i}$ for all even i with $i \leq r$ and $\dim \text{Ext}_{\mathbb{H} \rtimes \langle \theta \rangle}^i(X, X) = 0$ otherwise.

There is another application of the twisted Euler-Poincaré pairing for the deformation or complementary series of rigid modules.

Corollary 5.18. (c.f [4, Remark 4.6]) *For each $(J, U, \nu) \in \Xi$, set $X_\nu = X(J, U, \nu)$. Assume X_0 satisfies one of the three conditions in Proposition 2.11.*

- (1) *There exists a non-zero $\nu \in (V_J^\vee)^\perp$ such that $\text{Res}_W X_\nu \cong \text{Res}_W X_\nu^\theta$ only if X_0 is a rigid module.*
- (2) *There exists a non-zero $\nu \in (V_J^\vee)^\perp \cap V_0^\vee$ such that X_ν is $*$ -Hermitian only if X_0 is a rigid module.*

Proof. Suppose $\text{Res}_W X_\nu \cong \text{Res}_W X_\nu^\theta$ for some non-zero $\nu \in (V_J^\vee)^\perp$. Then by considering the central characters of the modules and using Theorem 2.19, $\text{EP}_{\mathbb{H}}^\theta(X_0, X_\nu) = 0$. Then by Theorem 4.11, $\text{EP}_{\mathbb{H}}^\theta(X_0, X_0) = 0$. Hence, X_0 is not a rigid module by Corollary 5.17. This proves (1). For (2), it follows from (1) and Proposition 2.11.

□

Example 5.19. The result for Theorem 5.15 is not true for other parabolically induced modules in general. For instance, consider \mathbb{H} of type A_2 . Take $J = \emptyset$. Let U be the one-dimensional trivial representation of $\overline{\mathbb{H}}_\emptyset = \mathbb{C}$ and let $X = X(\emptyset, U)$. Then $X(\emptyset, U)$

is an irreducible parabolically induced module of \mathbb{H} . Direct computation using Frobenius reciprocity shows

$$\dim \operatorname{Ext}_{\mathbb{H}}^i(X, X) = \begin{cases} 1 & \text{if } i = 0, 2 \\ 2 & \text{if } i = 1 \\ 0 & \text{if } i \geq 3 \end{cases}$$

Moreover, θ^* acts as an identity on $\operatorname{Ext}_{\mathbb{H}}^0(X, X)$, acts as the diagonal matrix $\operatorname{diag}(1, -1)$ on $\operatorname{Ext}_{\mathbb{H}}^1(X, X)$ and acts as -1 on $\operatorname{Ext}_{\mathbb{H}}^2(X, X)$.

6. SOLVABLE TEMPERED MODULES AND TWISTED ELLIPTIC SPACES

The goal of this section is to put or recollect some results in [10], [11], [12], [25] and [30] in the framework of twisted elliptic spaces.

6.1. Kazhdan-Lusztig model. In this section, let \mathbb{H} be the graded affine Hecke algebras associated to a crystallographic root datum (R, V, R^\vee, V^\vee) and an equal parameter function $k \equiv 1$. We also assume R spans V . Let \mathfrak{g} be the Lie algebra of the corresponding type. Let G be the simply-connected Lie group associated to \mathfrak{g} . According to the Kazhdan-Lusztig parametrization, there is a one-to-one correspondence between the set of irreducible tempered modules $X(e, \phi)$ with real central characters and the G -orbits of the set

$$\left\{ (e, \phi) : e \in \mathcal{N}, \phi \in \widehat{A(e)_0} \right\},$$

where \mathcal{N} is the set of nilpotent elements in \mathfrak{g} , $A(e)$ is the component group of e and $\widehat{A(e)_0}$ is the set of irreducible representation of the component group $A(e)$ that appears in the Springer correspondence.

We define \mathcal{N}_{sol} to be the set of nilpotent elements with a solvable centralizer in \mathfrak{g} . The interest for the set \mathcal{N}_{sol} can be found in [10], [4], [8] and [12]. We shall use the Bala-Carter symbols for the nilpotent orbits.

Definition 6.1. We say an irreducible tempered module $X(e, \phi)$ (with a real central character) is solvable if $e \in \mathcal{N}_{\text{sol}}$.

We need to use the following fact in the Kazhdan-Lusztig model [22, 6.2] (also see [25, 6.1a]):

Lemma 6.2. *Let e be a nilpotent element and let L be a Levi subgroup of G containing e . Let J be the subset of Δ associated to L and let $A_L(e)$ be the component group of e in L . Then for an $A_L(e)$ -representation ϕ , denote $U_J(e, \phi)$ the tempered \mathbb{H}_J -module associated to the pair (e, ϕ) in the Kazhdan-Lusztig model. Let $X_J(e, \phi) = U_J(e, \phi) \otimes \mathbb{C}_0$ be an $\mathbb{H}_J \cong \mathbb{H}_J \otimes S(V_J^\perp)$ -module. Then*

$$\operatorname{Ind}_{\mathbb{H}_J}^{\mathbb{H}} X_J(e, \phi) = X(e, \operatorname{Ind}_{A_L(e)}^{A(e)} \phi).$$

6.2. Dimension of twisted elliptic spaces. For Theorem 6.3 below, we apply the Kazhdan-Lusztig model to study the twisted elliptic spaces for non-trivial θ . Anyway, we shall use [30, Theorem 6.4] when θ is trivial and also apply some computations in [10]. Perhaps one may also apply [30, Theorem 6.4] or its line of argument to obtain Theorem 6.3 below in general.

Theorem 6.3. *Let \mathbb{H} be a graded affine Hecke algebra associated to a crystallographic root system and an arbitrary parameter function k . The dimension of $\text{Ell}_{\mathbb{H}}^{\theta}$ is equal to the number of θ -twisted elliptic conjugacy classes.*

Proof. For $\theta = \text{Id}$, it follows from [30, Theorem 6.4] (in more detail, one also has to apply [32, Proposition 6.4]). For $\theta \neq \text{Id}$, if $k_{\alpha} = 0$ for all $\alpha \in \Delta$, it is easy by Theorem 4.11. Thus we only consider the case that the parameter function $k_{\alpha} \neq 0$ for all $\alpha \in \Delta$. It is well-known that $\text{Res}_W X(e, \phi)$ (for all $e \in \mathcal{N}$ and $\phi \in \widehat{A(e)_0}$) spans the representation ring of W . Then the dimension of the spanning set of $\{\text{Res}_W X(e, \phi) \otimes S : e \in \mathcal{N}, \phi \in \widehat{A(e)_0}\}$ is equal to the number of twisted elliptic conjugacy classes. The last statement follows from a case-by-case analysis. The dimension of the spanning set follows from [10, Theorem 1.0.1]. The number of θ -twisted elliptic conjugacy classes is as follows:

A_n : number of partitions of n with distinct parts,

D_n (n odd) : number of partitions of n with odd number of parts, E_6 : 9.

Now by Theorem 4.11 and Proposition 4.15, we obtain that $\dim \text{Ell}_{\mathbb{H}}^{\theta}$ is equal to the number of θ -twisted elliptic conjugacy classes.

□

6.3. Description for twisted elliptic spaces.

Theorem 6.4. [12] *Let \mathbb{H} be a graded affine Hecke algebra associated to a crystallographic root system and an equal parameter function $k \equiv 1$. Then*

- (1) $\text{EP}_{\mathbb{H}}^{\theta}(X(e, \phi), X(e, \phi)) \neq 0$ for any $\phi \in \widehat{A(e)_0}$ if and only if $e \in \mathcal{N}_{\text{sol}}$.
- (2) $\text{EP}_{\mathbb{H}}^{\theta}(X(e, \phi), X(e', \phi')) = 0$ if e and e' are not in the same nilpotent orbit.
- (3) The set $\{[X(e, \phi)] : e \in \mathcal{N}_{\text{sol}}, \phi \in \widehat{A(e)_0}\}$ spans the θ -twisted elliptic space $\text{Ell}_{\mathbb{H}}^{\theta}$.

Proof. For (1) and (2), this is a direct consequence of Theorem 4.11 and results in [12, Theorem 1.1, Theorem 1.3]. For (3), it follows from (1) and the fact that $X(e, \phi)$ (for all nilpotent element e and all $\phi \in \widehat{A(e)_0}$) span the entire representation ring of W . From (1), we know that for $e \notin \mathcal{N}_{\text{sol}}$, $X(e, \phi)$ has a zero image in $\text{Ell}_{\mathbb{H}}^{\theta}$. Hence, the set in (3) spans the space $\text{Ell}_{\mathbb{H}}^{\theta}$.

We remark that for (2), one can also prove directly by considering the central characters of those modules. In more detail, the central character of $X(e, \phi)$ is $\frac{1}{2}h_e$, where $h_e \in V^{\vee}$ is the semisimple element in the \mathfrak{sl}_2 -triple $\{e, h_e, f\}$. If two nilpotent elements e and e' are not in the same nilpotent orbit, then the two elements h_e and $h_{e'}$ are not in the same W -orbit in V^{\vee} ([14, Theorem 2.2.4], [14, Theorem 3.2.14]).

In the case of type A_n , all solvable tempered modules are rigid (see the proof of Proposition 6.6 below). Thus for type A_n , (1) and (3) can also be obtained by Corollary 5.17 and a simple argument using Theorem 6.3 and using (2).

□

Remark 6.5. For arbitrary parameters, we expect some similar results as Theorem 6.4 can be obtained by considering tempered modules of solvable central characters. Here solvable central characters are in the sense of [8].

6.4. Relation between rigid modules and solvable tempered modules. We extend the notation of $X(e, \phi)$ to any $A(e)$ -representation ϕ : define

$$X(e, \phi) = \bigoplus_{\phi' \in \widehat{A(e)}_0} m_{\phi'} X(e, \phi'),$$

where $m_{\phi'} = \dim \operatorname{Hom}_{A(e)}(\phi', \phi)$.

Proposition 6.6. *Let \mathbb{H} be of type A_n , D_n (n odd) and E_6 . Let X be a parabolically induced tempered module with a real central character. Then X is solvable and irreducible if and only if X is rigid.*

Proof. This is a case-by-case analysis. To check which nilpotent orbits lie inside \mathcal{N}_{sol} , one may use the description of the centralizer of a nilpotent element in [7, Chapter 13] (also see [10]) (one may also verify by using the combinatorial criteria given in [8, Definition 1.1]).

For type A_n , a nilpotent element is in \mathcal{N}_{sol} if and only if the Jordan canonical form of e has blocks of distinct sizes. The Bala-Carter symbols for nilpotent elements in \mathcal{N}_{sol} coincide with the list for type A_n (Remark 5.3). Furthermore, for type A_n , all $X(e, \phi)$ for any $\phi \in \widehat{A(e)}_0$ are irreducible and hence the statement for type A_n is clear.

For type E_6 , a nilpotent element is in \mathcal{N}_{sol} if and only if the Bala-Carter symbol for the nilpotent element is of type E_6 , $E_6(a_1)$, $E_6(a_3)$, D_5 , $D_5(a_1)$, $A_4 + A_1$ and $D_4(a_1)$. The only type that does not appear in the classification of rigid modules is type $D_4(a_1)$. By Lemma 5.7, we only have to verify in the case that any irreducible tempered module associated to e of type $D_4(a_1)$ is not a parabolically induced module. Note that the corresponding component group $A(e)$ is S_3 and all representations of $A(e)$ appear in the Springer correspondence.

Let e be of type $D_4(a_1)$ and $\phi \in \widehat{A(e)}_0$. Suppose $X(e, \phi) = \operatorname{Ind}_{\mathbb{H}, J}^{\mathbb{H}} X_J(e, \phi')$ for some proper $J \subset \Delta$ and some $A_L(e)$ -representation ϕ' . Here we use the notation in Lemma 6.2. Note that J can only be of type D_5 or $D_4(a_1)$ and the component groups $A_L(e)$ of e for the Levi subgroups corresponding to D_5 and D_4 are S_2 , and 1 respectively, and hence $\operatorname{Ind}_{A_L(e)}^{A(e)} \phi'$ is not a single representation of S_3 . This contradicts the irreducibility of X . Hence $X(e, \phi)$ is not parabolically induced from some discrete series.

We now consider the case of D_n (n odd). In this case, a nilpotent element in $\mathfrak{so}(2n)$ is in \mathcal{N}_{sol} if and only if the partition of e contains only odd parts and each odd part has multiplicity at most 2. Then a similar analysis as in the case of E_6 will yield the result. In the analysis, we need the following description of the component group of (arbitrary)

nilpotent orbits for $\mathfrak{so}(2m)$ for both m odd and even (see for example [14, Chapter 6]): $A(e) = (\mathbb{Z}/2\mathbb{Z})^{\max(0, a-1)}$ if all odd parts have even multiplicity, $A(e) = (\mathbb{Z}/2\mathbb{Z})^{\max(0, a-2)}$ otherwise, where a is the number of distinct odd parts in the partition of e . We also need the component group of any nilpotent element in $\mathfrak{sl}(p)$ is trivial. Moreover, we also need the fact that for $e \in \mathcal{N}_{\text{sol}}$, all the representations of $A(e)$ appear in the Springer correspondence.

□

Remark 6.7. In type A_n and E_6 , solvable modules which are not elliptic are indeed rigid. However, in type D_n (n odd) with $n \geq 9$, if e is a nilpotent element corresponding to a partition satisfying the following three conditions:

- (1) e has no even parts, and
- (2) e has all odd parts with multiplicity 2, and
- (3) the number of distinct odd parts of e is at least 3,

then $X(e, \phi)$ is solvable, but neither rigid nor elliptic.

Remark 6.8. It is also possible to extend the condition of rigid modules to all solvable modules. We expect that an irreducible tempered module $X(e, \phi)$ with a real central character is solvable if and only if $X(e, \phi)$ is a submodule of a parabolically induced module $X(J, U)$ for some $(J, U, 0) \in \Xi$ such that

$$\text{card} \{w \in W : w(J) = J, w(U) = U\}$$

is equal to the sum of the square of the multiplicity of each irreducible submodule in $X(J, U)$.

6.5. Description of the radical of $\text{EP}_{\mathbb{H}}^{\theta}$. We end this paper with the following description of the radical:

Conjecture 6.9. *The radical $\text{rad}(\text{EP}_{\mathbb{H}}^{\theta})$ in $K_{\mathbb{H}}^{\theta}$ is equal to the image of*

$$\bigoplus_{J \in \mathcal{J}^{\theta}} \text{Ind}_{\mathbb{H}_J \rtimes \langle \theta \rangle}^{\mathbb{H} \rtimes \langle \theta \rangle} K_{\mathbb{C}}(\mathbb{H}_J \rtimes \langle \theta \rangle).$$

When $\theta = \text{Id}$, it is known to be true from [30, Theorem 6.4]. It is also possible to apply [30, Theorem 6.4] or its proof for the conjecture in general. For non-trivial θ , it is not too hard to verify directly for type A_n and E_6 , but it seems more effort has to be done for type D_n (n odd).

REFERENCES

- [1] J. Adler and D. Prasad, *Extensions of representations of p -adic groups*, Nagoya Math. J. **208** (2012), 171-199.
- [2] J. Arthur, *On elliptic tempered characters*, Acta. Math. **171** (1993), 73-138.
- [3] D. Barbasch and D. Ciubotaru, *Hermitian form for affine Hecke algebras*, arxiv: 1312.3316v1 [math.RT].
- [4] D. Barbasch, D. Ciubotaru and P. Trapa, *Dirac cohomology for graded affine Hecke algebras*, Acta. Math. **209** (2) (2012), 197-227.
- [5] D. Barbasch and A. Moy, *A unitarity criterion for p -adic groups*, Invent. Math. **98** (1989), 19-37.
- [6] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, **94** (1980), Princeton University Press (Princeton, NJ).

- [7] R. Carter, *Finite groups of Lie type. Conjugacy classes and complex and characters*, Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc. New York, 1985.
- [8] K. Y. Chan, *Spin representations of real reflection groups of non-crystallographic types*, J. of Algebra **379** (2013), 333-354.
- [9] K. Y. Chan, *Note on elliptic representation characters for affine Hecke algebras and limits of discrete series*, in preparation.
- [10] D. Ciubotaru, *Spin representations of Weyl groups and the Springer correspondence*, J. Reine Angew. Math. **671** (2012), 199-222.
- [11] D. Ciubotaru and X. He, *The cocenter of graded affine Hecke algebra and the density theorem*, arxiv:1208.0914 [math.RT].
- [12] D. Ciubotaru and X. He, *Green polynomials of Weyl groups, elliptic pairings, and the extended Dirac index*, arXiv:1303.6806 [math.RT].
- [13] D. Ciubotaru and P. Trapa, *Characters of Springer representations on elliptic classes*, Duke Math. J. (2) **162** (2013), 201-223.
- [14] D. H. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold, New York, 1985.
- [15] P. Delorme and E. M. Opdam, *The Schwartz algebra of an affine Hecke algebra*, J. Reine Angew. Math., **625** (2008), 59-114.
- [16] P. Delorme and E. M. Opdam, *Analytic R-groups for affine Hecke algebras*, J. Reine Angew. Math., **658** (2011), 133-172.
- [17] S. Evens, I. Mirković, *Fourier transform and the Iwahori-Matsumoto involution*, Duke Math. J. **86** (1997), no. 3, 435-464.
- [18] M. Geck, S. Kim, G. Pfeiffer, *Minimal length elements in twisted conjugacy classes of finite Weyl groups*, J. Algebra **229** (2000), 570-600.
- [19] G. J. Heckman and E. M. Opdam, *Yang's system of particles and Hecke algebras*, Math. Ann. **145** (1997), 139-173.
- [20] J. S. Huang and P. Pandžić, *Dirac operators in representation theory*, Mathematics: Theory & applications (2006), Birkhäuser.
- [21] J. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Univ. Press, 1990.
- [22] D. Kazhdan and G. Lusztig, *Proof for the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math., **87** (1987), 153-215.
- [23] C. Krilloff, A. Ram, *Representations of graded Hecke algebras*, Represent. Theory **6** (2002), 31-69.
- [24] G. Lusztig, *Affine Hecke algebras and their graded versions*, J. Amer. Math. Soc. **2** (1989), 599-635.
- [25] M. Reeder, *Euler-Poincaré pairings and elliptic representations of Weyl groups and p-adic groups*, Compositio Math. **129** (2001), 149-181.
- [26] E. M. Opdam, *On the spectral decomposition of affine Hecke algebras*, J. Inst. Math. Jussieu **3** (2004), no. 4, 531-648.
- [27] E.M. Opdam, *Harmonic analysis for certain representations of graded Hecke algebras*, Acta. Math. **175** (1995), no. 1, 75-121.
- [28] E.M. Opdam and M. Solleveld, *Homological algebra for affine Hecke algebras*, Adv. in Math. **220** (2009), 1549-1601.
- [29] E.M. Opdam and M. Solleveld, *Discrete series characters for affine Hecke algebras and their formal degrees*, Acta Mathematica. **205** (2010), 105-187.
- [30] E.M. Opdam and M. Solleveld, *Extensions of tempered representations*, Geometric And Functional Analysis **23** (2013), 664-714.
- [31] P. Schneider and U. Stuhler, *Representation theory and sheaves on the Bruhat-Tits building*, Publ. Math. Inst. Hautes Etudes Sci. **85** (1997), 97-191.
- [32] M. Solleveld, *Parabolically induced representations of graded Hecke algebras*, Algebras and Representation Theory **152** (2012), 233-271.

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